CH 3 homework solutions

3.3

The given signal is

$$\begin{aligned} x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \qquad a_2 = a_{-2} = \frac{1}{2}, \qquad a_5 = a_{-5}^* = -2j$$

3.7

Since $x(t)$ is real and odd (clue 1), its Fourier series coefficients a_k are purely imaginary and odd (See Table 3.1). Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. Also, since it is given that $a_k = 0$ for $|k| > 1$, the only unknown Fourier series coefficients are a_1 and a_{-1}. Using Parseval's relation,

$$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

for the given signal we have

$$\frac{1}{2}\int_0^2 |x(t)|^2 dt = \sum_{k=-1}^{1} |a_k|^2.$$

Using the information given in clue (4) along with the above equation,

$$|a_1|^2 + |a_{-1}|^2 = 1 \qquad \Rightarrow \qquad 2|a_1|^2 = 1$$

Therefore,

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}j} \qquad \text{or} \qquad a_1 = -a_{-1} = -\frac{1}{\sqrt{2}j}$$

The two possible signals which satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} - \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = -\sqrt{2}\sin(\pi t)$$

and

$$x_2(t) = -\frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} + \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = \sqrt{2}\sin(\pi t)$$

3.10

Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundemental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundemental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \xleftrightarrow{FS} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \xleftrightarrow{FS} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \xleftrightarrow{FS} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \xleftrightarrow{FS} a_{-k} e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \xleftrightarrow{FS} a_k e^{jk(2\pi/T_1)} + a_{-k} e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k}(a_k + a_{-k})$$

3.11

Since the Fourier series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$. Furthermore, since $x[n]$ is real and even, a_k is also real and even. Therefore, $a_1 = a_{-1} = 5$. We are also given that

$$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50.$$

Using Parseval's relation,

$$\begin{aligned}
\sum_{k=<N>}|a_k|^2 &= 50 \\
\sum_{k=-1}^{8}|a_k|^2 &= 50 \\
|a_{-1}|^2 + |a_1|^2 + a_0^2 + \sum_{k=2}^{8}|a_k|^2 &= 50 \\
a_0^2 + \sum_{k=2}^{8}|a_k|^2 &= 0
\end{aligned}$$

Therefore, $a_k = 0$ for $k = 2, \cdots, 8$. Now using the synthesis eq.(3.94), we have

$$\begin{aligned}
x[n] &= \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^{8} a_k e^{j\frac{2\pi}{10}kn} \\
&= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n} \\
&= 10\cos(\frac{\pi}{5}n)
\end{aligned}$$

3.13

Let us first evaluate the Fourier series coefficients of $x(t)$. Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd. Therefore, $a_0 = 0$. Now,

$$\begin{aligned}
a_k &= \frac{1}{8}\int_0^8 x(t)e^{-j(2\pi/8)kt}dt \\
&= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}dt
\end{aligned}$$

$$= \frac{1}{j\pi k}[1 - e^{-j\pi k}]$$

Clearly, the above expression evaluates to zero for all even values of k. Therefore,

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \cdots \\ \frac{2}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by (see Section 3.8)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Since a_k is non zero only for odd values of k, we need to evaluate the above summation only for odd k. Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

is always zero for odd values of k. Therefore,

$$y(t) = 0.$$

3.14

The signal $x[n]$ is periodic with period $N = 4$. Its Fourier series coefficients are

$$\begin{aligned} a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}, \quad \text{for all } k \end{aligned}$$

From the results presented in Section 3.8, we know that the output $y[n]$ is given by

$$\begin{aligned} y[n] &= \sum_{k=0}^{3} a_k H(e^{j(2\pi/4)k}) e^{jk(2\pi/4)n} \\ &= \tfrac{1}{4}H(e^{j0})e^{j0} + \tfrac{1}{4}H(e^{j(\pi/2)})e^{j(\pi/2)} \\ &\quad + \tfrac{1}{4}H(e^{j(3\pi/2)})e^{j(3\pi/2)} + \tfrac{1}{4}H(e^{j(\pi)})e^{j(\pi)} \end{aligned} \qquad \text{(S3.14-1)}$$

From the given information, we know that $y[n]$ is

$$\begin{aligned} y[n] &= \cos(\frac{5\pi}{2}n + \frac{\pi}{4}) \\ &= \cos(\frac{\pi}{2}n + \frac{\pi}{4}) \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{-j(\frac{\pi}{2}n + \frac{\pi}{4})} \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{j(3\frac{\pi}{2}n - \frac{\pi}{4})} \end{aligned}$$

Comparing this with eq. (S3.14-1), we have

$$H(e^{j0}) = H(e^{j\pi}) = 0$$

and

$$H(e^{j\frac{\pi}{2}}) = 2e^{j\frac{\pi}{4}}, \qquad \text{and} \qquad H(e^{3j\frac{\pi}{2}}) = 2e^{-j\frac{\pi}{4}}$$