CH5 ans

5.4

The inverse Fourier transform of $x[n]$ is given as,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

(1) $X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\left\{2\pi\delta(\omega-2\pi k)+\pi\delta(w-\frac{\pi}{2}-2\pi k)+\pi\delta(\omega+\frac{\pi}{2}-2\pi k)\right\}$

$$x_1[n] = \frac{1}{2\pi}\int_0^{2\pi} X_1(e^{j\omega})e^{j\omega n}d\omega$$

within the interval $0\le\omega\le 2\pi$, $X_1(e^{j\omega}) = 2\pi\delta(\omega)+\pi\delta(w-\frac{\pi}{2})+\pi\delta(\omega+\frac{\pi}{2})$.
Therefore,

$$\begin{aligned}
x_1[n] &= \frac{1}{2\pi}\int_0^{2\pi}\left\{2\pi\delta(\omega)+\pi\delta(w-\frac{\pi}{2})+\pi\delta(\omega+\frac{\pi}{2})\right\}e^{j\omega n}d\omega \\
x_1[n] &= \frac{1}{2\pi}\left[\int_0^{2\pi}2\pi\delta(\omega)e^{j\omega n}d\omega+\int_0^{2\pi}\pi\delta(w-\frac{\pi}{2})e^{j\omega n}d\omega+\int_0^{2\pi}\pi\delta(\omega+\frac{\pi}{2})e^{j\omega n}d\omega\right] \\
x_1[n] &= \frac{1}{2\pi}\left[2\pi e^{j(0)n}+\pi e^{j(\pi/2)n}+\pi e^{j(-\pi/2)n}\right] \qquad \left(\because \int_{-\infty}^{\infty}x(t)\delta(t-t_0)dt = x(t_0)\right) \\
x_1[n] &= \frac{2\pi+\pi\left(e^{j(\pi/2)n}+e^{-j(\pi/2)n}\right)}{2\pi} \\
x_1[n] &= 1+\frac{\left(e^{j(\pi/2)n}+e^{-j(\pi/2)n}\right)}{2} \qquad \left(\because \cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}\right) \\
x_1[n] &= 1+\cos\left(\frac{\pi n}{2}\right)
\end{aligned}$$

(2) $X_2(e^{j\omega}) = \begin{cases} 2j, & 0<\omega\le\pi \\ -2j, & -\pi<\omega\le 0\end{cases}$

$$\begin{aligned}
x_2[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}X_2(e^{j\omega})e^{j\omega n}d\omega \\
x_2[n] &= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-2j)e^{j\omega n}d\omega+\int_0^{\pi}(2j)e^{j\omega n}d\omega\right] \\
x_2[n] &= \frac{2j}{2\pi}\left[\int_0^{\pi}e^{j\omega n}d\omega-\int_{-\pi}^{0}e^{j\omega n}d\omega\right] \\
x_2[n] &= \frac{2j}{2\pi}\left[\left\{\frac{e^{j\omega n}}{jn}\right\}_0^{\pi}-\left\{\frac{e^{j\omega n}}{jn}\right\}_{-\pi}^{0}\right] \\
x_2[n] &= \frac{2j}{2\pi}\left[\frac{e^{j\pi n}}{jn}-\frac{e^{j(0)n}}{jn}-\frac{e^{j(0)n}}{jn}+\frac{e^{-j\pi n}}{jn}\right] \\
x_2[n] &= \frac{1}{n\pi}\left[e^{j\pi n}+e^{-j\pi n}-2\right] = \frac{2}{n\pi}\left[\frac{e^{j\pi n}+e^{-j\pi n}}{2}-1\right] \\
x_2[n] &= \frac{2}{n\pi}\left[\cos(n\pi)-1\right] \qquad \left(\because \cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}\right) \\
x_2[n] &= \frac{-4}{n\pi}\sin^2\left(\frac{n\pi}{2}\right) \qquad \left(\because \cos(2\theta) = 1-2\sin^2\theta\right)
\end{aligned}$$

5.6

We know that Fourier transform pair is:

$$x[n] \overset{\text{FT}}{\longleftrightarrow} X(e^{j\omega})$$

a)

$$x_1[n] = x[1-n] + x[-1-n]$$

Apply time shifting property of the Fourier transform:

$$x[n-n_0] \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$x[n-1] \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega} X(e^{j\omega})$$

$$x[n+1] \overset{\text{FT}}{\longleftrightarrow} e^{j\omega} X(e^{j\omega})$$

And now apply time reversal property of Fourier transform:

$$x[-n] \overset{\text{FT}}{\longleftrightarrow} X(e^{-j\omega})$$

$$x[-n+1] \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega} X(e^{-j\omega})$$

$$x[-n-1] \overset{\text{FT}}{\longleftrightarrow} e^{j\omega} X(e^{-j\omega})$$

and finally, apply linearity property of FT:

$$x_1[n] \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega} X(e^{-j\omega}) + e^{j\omega} X(e^{-j\omega})$$

$$x_1[n] \overset{\text{FT}}{\longleftrightarrow} 2X(e^{-j\omega}) \cdot \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$\boxed{x_1[n] \overset{\text{FT}}{\longleftrightarrow} 2\cos\omega X(e^{-j\omega})}$$

b)

$$x_2[n] = \frac{x^*[-n] + x[n]}{2} = \frac{1}{2}x^*[-n] + \frac{1}{2}x[n]$$

Use conjugation property of Fourier transform:

$$x^*[n] \overset{\text{FT}}{\longleftrightarrow} X^*(e^{-j\omega})$$

Also use time reversal property of FT:

$$x^*[-n] \overset{\text{FT}}{\longleftrightarrow} X^*(e^{j\omega})$$

and finnaly, with linearity property we obtain $x_2[n]$:

$$\boxed{x_2[n] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2}\left(X^*(e^{j\omega}) + X(e^{j\omega})\right) = \text{Re}\{X(e^{j\omega})\}}$$

c)

$$x_3[n] = (n-1)^2 x[n] = n^2 x[n] - 2nx[n] + x[n]$$

Apply the differentiation in frequency property of Fourier transform:

$$nx[n] \overset{\text{FT}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

$$n^2 x[n] \overset{\text{FT}}{\longleftrightarrow} j^2 \frac{d^2 X(e^{j\omega})}{d\omega^2}$$

Use linearity property of FT:

$$\boxed{x_3[n] \overset{\text{FT}}{\longleftrightarrow} -\frac{d^2 X(e^{j\omega})}{d\omega^2} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})}$$

5.8

Consider the signal:

$$x_1[n] = \begin{cases} 1; & |n| \le 1 \\ 0; & |n| > 1 \end{cases}$$

Fourier transform of the signal $x_1[n]$ is:

$$x_1[n] \overset{\text{FT}}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{\sin(3\omega/2)}{\sin(\omega/2)}$$

Now use the accumulation property of Fourier transform:

$$\sum_{k=-\infty}^{n} x_1[k] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X_1(e^{j\omega}) + \pi X_1(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Now consider the range of the original signal $-\pi < \omega < \pi$, where the sum $\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ is:

$$\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) = \delta(\omega)$$

$$\sum_{k=-\infty}^{n} x_1[k] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X_1(e^{j\omega}) + \pi X_1(e^{j0})\delta(\omega)$$

Now determine the value of $X_1(e^{j0})$:

$$X_1(e^{j0}) = \lim_{\omega \to 0} \frac{\sin(3\omega/2)}{\sin(\omega/2)} = 3 \lim_{\omega \to 0} \frac{\frac{\sin(3\omega/2)}{\frac{3}{2}}}{\frac{\sin(1\omega/2)}{\frac{1}{2}}} = 3$$

Now we have:

$$\sum_{k=-\infty}^{n} x_1[k] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X_1(e^{j\omega}) + 3\pi\delta(\omega)$$

we are for $2\pi\delta(\omega)$ 'short' on the RHS of equation, but in the range $-\pi < \omega < \pi$:

$$1 \overset{\text{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$$

So, we can write for $x[n]$:

$$x[n] = 1 + \sum_{k=-\infty}^{n} x_1[k] \overset{\mathrm{FT}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X_1(e^{j\omega}) + 5\pi\delta(\omega)$$

Then the signal $x[n]$ is:

$$x[n] = \begin{cases} 1; \ n \leq -2 \\ n+3; \ -1 \leq n \leq 1 \\ 4; \ n > 2 \end{cases}$$

5.10

Let's start from Fourier transform pair:

$$x[n] \overset{\mathrm{FT}}{\longleftrightarrow} X(e^{j\omega})$$

The conjugate symmetry for real signals property in combination with conjugation property is:

$$Od\{x[n]\} \overset{\mathrm{FT}}{\longleftrightarrow} j\,\mathrm{Im}\{X(e^{j\omega})\} \tag{1}$$

consider the fact about the function $x[n]$ to determine RHS of the equation:

$$\mathrm{Im}\{X(e^{j\omega})\} = \sin\omega - \sin 2\omega$$

$$j\,\mathrm{Im}\{X(e^{j\omega})\} = j\sin\omega - j\sin 2\omega$$

$$j\,\mathrm{Im}\{X(e^{j\omega})\} = j\frac{e^{j\omega}-e^{-j\omega}}{2j} - j\frac{e^{j2\omega}-e^{-j2\omega}}{2j}$$

$$j\,\mathrm{Im}\{X(e^{j\omega})\} = \frac{1}{2}(e^{j\omega} - e^{-j\omega} - e^{j2\omega} + e^{-j2\omega})$$

and the LHS of the equation (1) is:

$$Od\{x[n]\} = F^{-1}\{j\,\mathrm{Im}\{X(e^{j\omega})\}\}$$

$$F^{-1}\{j\,\mathrm{Im}\{X(e^{j\omega})\}\} = \frac{1}{2}(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$$

And from the conjugate symmetry, the odd part of the function is:

$$Od\{x[n]\} = \frac{x[n]-x[-n]}{2}$$

and from the fact that $x[n>0]=0$ we have:

$$x[n] = 2Od\{x[n]\}$$

$$x[n] = \delta[n+1] - \delta[n+2]$$

And finally we have to determine $x[0]$, using Parseval's relation:

$$\frac{1}{2\pi}\int -\pi^{\pi}|X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty}|x[n]|^2$$

$$\frac{1}{2\pi}\int -\pi^{\pi}|X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{-1}|x[n]|^2 + |x[0]|^2 + \sum_{n=1}^{\infty}|x[n]|^2$$

since $x[n>0]=0$

$$\frac{1}{2\pi}\int -\pi^{\pi}|X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{-1}|x[n]|^2 + |x[0]|^2$$

Substitute the value from 4. condition in previous equation:

$$3 = \sum_{n=-\infty}^{-1} |x[n]|^2 + |x[0]|^2$$

$$3 = 1^2 + (-1)^2 + |x[0]|^2$$

since $x[0]$ is positive:

$$x[0] = 1$$

Therefore, the signal $x[n]$ is:

$$\boxed{x[n] = \delta[n+1] - \delta[n+2] + \delta[n]}$$

5.13

Consider the signal

$$x_1[n] = \left(\frac{\sin \frac{\pi}{4} n}{\pi n}\right).$$

From Table 5.2, we obtain the Fourier transform of $x_1[n]$ to be

$$X_1(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}.$$

The plot of $X_1(e^{j\omega})$ is as shown in the Figure S5.12. Now consider the signal $x_2[n] = (x_1[n])^2$. Using the multiplication property (Table 5.1, Property 5.5), we obtain the Fourier tranform of $x_2[n]$ to be

$$X_2(e^{j\omega}) = (1/2\pi)[X_1(e^{j\omega}) * X_1(e^{j\omega})].$$

This is plotted in the Figure S5.12.

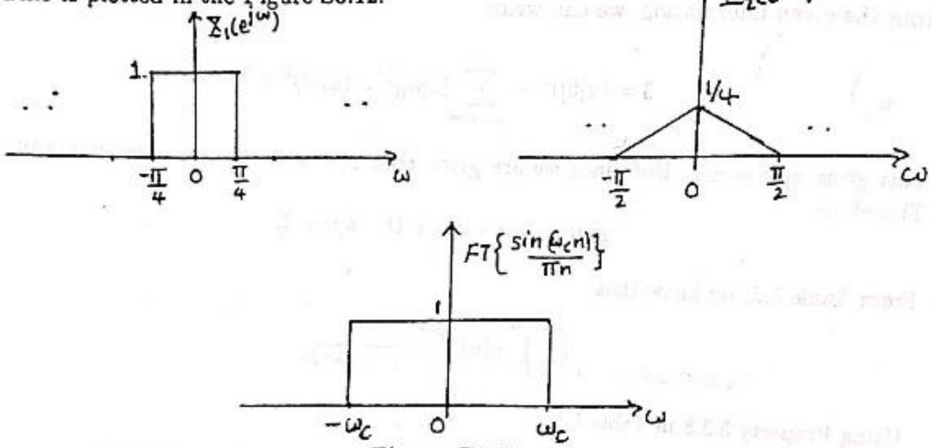

Figure S5.12

From Figure S5.12 it is clear that $X_2(e^{j\omega})$ is zero for $|\omega| > \pi/2$. By using the convolution property (Table 5.1, Property 5.4), we note that

$$Y(e^{j\omega}) = X_2(e^{j\omega}) \mathcal{FT}\left\{\frac{\sin(\omega_c n)}{\pi n}\right\}.$$

The plot of $\mathcal{FT}\left\{\frac{\sin(\omega_c n)}{\pi n}\right\}$ is shown in Figure S5.12. It is clear that if $Y(e^{j\omega}) = X_2(e^{j\omega})$, then $(\pi/2) \le \omega_c \le \pi$.

5.14

When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore,

$$h[n] = h_1[n] + h_2[n].$$

Using the linearity property (Table 5.1, Property 5.3.2),

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Given that $h_1[n] = (1/3)^n u[n]$, we obtain

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

Therefore,

$$H_2(e^{j\omega}) = \frac{-12+5^{-j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}} - \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}.$$

Taking the inverse Fourier transform,

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n].$$

5.20

Knowing that

$$\left(\frac{1}{2}\right)^{|n|} \overset{FT}{\longleftrightarrow} \frac{1-\frac{1}{4}}{1-\cos\omega+\frac{1}{4}} = \frac{3}{5-4\cos\omega},$$

we may use the Fourier transform analysis equation to write

$$\frac{3}{5-4\cos\omega} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega n}$$

Putting $\omega = -2\pi t$ in this equation, and replacing the variable n by the variable k

$$\frac{1}{5-4\cos(2\pi t)} = \sum_{k=-\infty}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{|k|} e^{j2\pi kt}.$$

By comparing this with the continuous-time Fourier series synthesis equation, it is ir mediately apparent that $a_k = \frac{1}{3}\left(\frac{1}{2}\right)^{|k|}$ are the Fourier series coefficients of the sign $1/(5-4\cos(2\pi t))$.

5.21

a) Take the Fourier transform of given equation (on both sides, use Table 5.2.):

$$Y(e^{j\omega})\left(1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}\right) = X(e^{j\omega})$$

The frequency response $H(e^{j\omega})$ of given system S is:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{3}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{3}e^{-j\omega}\left(1-\frac{1}{2}e^{-j\omega}\right)} \\ &= \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} \end{aligned}$$

b) Now use partial fraction expansion on $H(e^{j\omega})$:

$$H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

$$A\left(1+\frac{1}{3}e^{-j\omega}\right) + B\left(1-\frac{1}{2}e^{-j\omega}\right) = 1$$

$$A + B = 1$$

$$A\cdot\frac{1}{3} - B\cdot\frac{1}{2} = 0$$

$$A = \frac{3}{5} \text{ and } B = \frac{2}{5}$$

$$H(e^{j\omega}) = \frac{\frac{3}{5}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{2}{5}}{1+\frac{1}{3}e^{-j\omega}}$$

Now using table 5.2. and inverse Fourier transform, we can determine $h[n]$:

$$\boxed{h[n] = \left[\frac{3}{5}\left(\frac{1}{2}\right)^n + \frac{2}{5}\left(-\frac{1}{3}\right)^n\right]u[n]}$$

5.32

$$X(e^{jw}) = e^{j2w}(1-e^{-j3w})$$

$$= e^{j2w} - e^{-jw}$$

$$e^{-jwn_0} \overset{FT}{\longleftrightarrow} \delta[n-n_0]$$

$$x[n] = \delta[n+2] - \delta[n-1] \quad \#$$

5.35

a) Let $y[n]$ be the output of the given LTI system. Then its Fourier transform pair is (we're using convolution property of discrete time Fourier transform):

$$y[n] = x[n] * h[n] \overset{\text{FT}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

According to table 5.2. (basic discrete-time Fourier transform pairs), the Fourier transform pair for $h[n]$ is:

$$a^n u[n] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

for $a = 1/2$, we have for $h[n]$:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \overset{\text{FT}}{\longleftrightarrow} H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

i) For $x[n] = \left(\frac{3}{4}\right)^n u[n]$, Fourier transform pair is according to table 5.2. (basic discrete-time Fourier transform pairs)

$$a^n u[n] \stackrel{\text{FT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

for $a = 3/4$, we have for $x[n]$:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \stackrel{\text{FT}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{1 - \frac{3}{4}e^{-j\omega}}$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\ &= \frac{1}{1 - \frac{3}{4}e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \\ &= \frac{A}{1 - \frac{3}{4}e^{-j\omega}} + \frac{B}{1 - \frac{1}{2}e^{-j\omega}} \end{aligned}$$

Where coefficients A and B are:

$$A\left(1 - \frac{1}{2}e^{-j\omega}\right) + B\left(1 - \frac{3}{4}e^{-j\omega}\right) = 1$$

$$A + B = 1 \text{ and } A\frac{1}{2}e^{-j\omega} + B\frac{3}{4}e^{-j\omega} = 0$$

$$A = 3 \text{ and } B = -2$$

$$Y(e^{j\omega}) = \frac{3}{1 - \frac{3}{4}e^{-j\omega}} - \frac{2}{1 - \frac{1}{2}e^{-j\omega}}$$

Now apply the inverse Fourier transform (using the same basic discrete-time Fourier transform pair):

$$\frac{1}{1 - ae^{-j\omega}} \stackrel{\text{FT}}{\longleftrightarrow} a^n u[n]$$

Then signal $y[n]$ is:

$$\boxed{y[n] = \left(3 \cdot \left(\frac{3}{4}\right)^n - 2 \cdot \left(\frac{1}{2}\right)^n\right) u[n]}$$

ii) For $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$, Fourier transform pair is according to table 5.2. (basic discrete-time Fourier transform pairs)

$$(n+1)a^n u[n] \stackrel{\text{FT}}{\longleftrightarrow} \frac{1}{(1 - ae^{-j\omega})^2}$$

for $a = 1/4$, we have for $x[n]$:

$$x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n] \stackrel{\text{FT}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2}$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}} \\
&= \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{C}{1-\frac{1}{2}e^{-j\omega}}
\end{aligned}$$

Where coefficients A, B and C are:

$$A\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right) + B\left(1-\frac{1}{2}e^{-j\omega}\right) + C\left(1-\frac{1}{4}e^{-j\omega}\right) = 1$$

for $e^{-j\omega} = 2$, we have:

$$\frac{C}{4} = 1 \rightarrow C = 4$$

for $e^{-j\omega} = 4$, we have:

$$-B = 1 \rightarrow B = -1$$

and from term:

$$A + B + C = 1 \rightarrow A = -2$$

$$Y(e^{j\omega}) = -\frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{4}{1-\frac{1}{2}e^{-j\omega}}$$

Now apply the inverse Fourier transform using following basic discrete-time Fourier transform pairs:

$$\frac{1}{\left(1-ae^{-j\omega}\right)^2} \stackrel{\text{FT}}{\longleftrightarrow} (n+1)a^n u[n]$$

$$\frac{1}{1-ae^{-j\omega}} \stackrel{\text{FT}}{\longleftrightarrow} a^n u[n]$$

Then signal $y[n]$ is:

$$\boxed{y[n] = \left(4 \cdot \left(\frac{1}{2}\right)^n - 2 \cdot \left(\frac{1}{4}\right)^n - (n+1)\left(\frac{1}{4}\right)^n\right) u[n]}$$

iii) For $x[n] = (-1)^n = \left(e^{j\pi}\right)^n = e^{j\pi n}$, Fourier transform pair is according to table 5.2. (basic discrete-time Fourier transform pairs)

$$e^{j\omega_0 n} \stackrel{\text{FT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2\pi k\right)$$

for $\omega_0 = \pi$, we have for $x[n]$:

$$x[n] = e^{j\pi n} \stackrel{\text{FT}}{\longleftrightarrow} X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \pi - 2\pi k\right) = 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - (2k+1)\pi\right)$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega-(2k+1)\pi\right) \\
&= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) \cdot \delta\left(\omega-(2k+1)\pi\right) \\
&= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{1}{1-\frac{1}{2}e^{-j(2k+1)\pi}}\right) \cdot \delta\left(\omega-(2k+1)\pi\right)
\end{aligned}$$

since $2k+1$ is obviously odd number, we have that:

$$1-\frac{1}{2}e^{-j(2k+1)\pi} = 1-\frac{1}{2}\left(e^{-j\pi}\right)^{2k+1} = 1-\frac{1}{2}\cdot(-1)^{2k+1} = 1-\frac{1}{2}\cdot(-1) = \frac{3}{2}$$

$$Y(e^{j\omega}) = \frac{2}{3}\cdot 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega-(2k+1)\pi\right)$$

Now apply the inverse Fourier transform using following basic discrete-time Fourier transform pair:

$$2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega-\omega_0-2\pi k\right) \xleftrightarrow{\text{FT}} e^{j\omega_0 n}$$

Then signal $y[n]$ is:

$$\boxed{y[n] = \frac{2}{3}e^{j\pi n} = \frac{2}{3}\left(e^{jn}\right)^{\pi} = (-1)^n\frac{2}{3}}$$

b) Let $y[n]$ be the output of the given LTI system. Then its Fourier transform pair is (we're using convolution property of discrete time Fourier transform):

$$y[n] = x[n] * h[n] \xleftrightarrow{\text{FT}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Fourier transform for $h[n]$ is:

$$\begin{aligned}
h[n] &= \left[\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right)\right] u[n] \\
&= \left[\left(\frac{1}{2}\right)^n \frac{e^{j\frac{\pi n}{2}}+e^{-j\frac{\pi n}{2}}}{2}\right] u[n] \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\right)^n e^{j\frac{\pi n}{2}} + \left(\frac{1}{2}\right)^n e^{-j\frac{\pi n}{2}}\right] u[n] \\
&= \frac{1}{2}\left(\frac{1}{2}e^{j\frac{\pi}{2}}\right)^n + \frac{1}{2}\left(\frac{1}{2}e^{-j\frac{\pi}{2}}\right)^n
\end{aligned}$$

Now refer to the table 5.2., basic discrete-time Fourier transform pairs, and use:

$$\frac{1}{1-ae^{-j\omega}} \xleftrightarrow{\text{FT}} a^n u[n]$$

then:

$$H(e^{j\omega}) = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{j\frac{\pi}{2}}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}e^{-j\omega}}$$

since $e^{-j\frac{\pi}{2}} = -j$ and $e^{j\frac{\pi}{2}} = j$ we have:

$$H(e^{j\omega}) = \frac{\frac{1}{2}}{1-j\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1+j\frac{1}{2}e^{-j\omega}}$$

i) For $x[n] = \left(\frac{1}{2}\right)^n u[n]$, Fourier transform pair is according to table 5.2. (basic discrete-time Fourier transform pairs)

$$a^n u[n] \xleftrightarrow{\text{FT}} \frac{1}{1 - ae^{-j\omega}}$$

for $a = 1/2$, we have for $h[n]$:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\text{FT}} X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \cdot \left(\frac{\frac{1}{2}}{1 - j\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1 + j\frac{1}{2}e^{-j\omega}} \right) \\
&= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \cdot \frac{\frac{1}{2}\left(1 + j\frac{1}{2}e^{-j\omega} + 1 - j\frac{1}{2}e^{-j\omega}\right)}{\left(1 - j\frac{1}{2}e^{-j\omega}\right)\left(1 + j\frac{1}{2}e^{-j\omega}\right)} \\
&= \frac{1}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - j\frac{1}{2}e^{-j\omega}\right)\left(1 + j\frac{1}{2}e^{-j\omega}\right)} \\
&= \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - j\frac{1}{2}e^{-j\omega}} + \frac{C}{1 + j\frac{1}{2}e^{-j\omega}}
\end{aligned}$$

Where coefficients A, B and C are, for $e^{-j\omega} = 2$, we have:

$$A = \frac{1}{2}$$

for $e^{-j\omega} = \frac{2}{j}$, we have:

$$B = \frac{j}{2(j-1)}$$

for $e^{-j\omega} = -\frac{2}{j}$, we have:

$$C = \frac{j}{2(j+1)}$$

$$Y(e^{j\omega}) = \frac{1}{2}\frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{j}{2(j-1)}\frac{1}{1 - j\frac{1}{2}e^{-j\omega}} + \frac{j}{2(j+1)}\frac{1}{1 + j\frac{1}{2}e^{-j\omega}}$$

Now apply the inverse Fourier transform (using the same basic discrete-time Fourier transform pair):

$$\frac{1}{1 - ae^{-j\omega}} \xleftrightarrow{\text{FT}} a^n u[n]$$

Then signal $y[n]$ is:

$$\boxed{y[n] = \left(\frac{1}{2} \cdot \left(\frac{1}{2}\right)^n + \frac{j}{2(j-1)}\left(\frac{j}{2}\right)^n + \frac{j}{2(j+1)}\left(-\frac{j}{2}\right)^n\right) u[n]}$$

ii) For $x[n] = \cos \pi n/2$, Fourier transform pair is according to table 5.2. (basic discrete-time Fourier transform pairs)

$$\cos \omega_0 n \stackrel{\mathrm{FT}}{\longleftrightarrow} \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l))$$

for $\omega_0 = \pi/2$, we have for signal $x[n]$:

$$x[n] = \cos \pi n/2 \stackrel{\mathrm{FT}}{\longleftrightarrow} X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l))$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \left(\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l))\right) \cdot \left(\frac{\frac{1}{2}}{1 - j\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1 + j\frac{1}{2}e^{-j\omega}}\right) \\
&= \left(\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l))\right) \cdot \frac{\frac{1}{2}\left(1 + j\frac{1}{2}e^{-j\omega} + 1 - j\frac{1}{2}e^{-j\omega}\right)}{\left(1 - j\frac{1}{2}e^{-j\omega}\right)\left(1 + j\frac{1}{2}e^{-j\omega}\right)} \\
&= \left(\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l))\right) \cdot \left(\frac{1}{1 + \frac{e^{-2j\omega}}{4}}\right) \\
&= \left(\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l))\right) \cdot \left(\frac{4}{4 + e^{-2j\omega}}\right) \\
&= \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-2j\left(\frac{\pi}{2} + 2\pi l\right)}}\right) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-2j\left(\frac{\pi}{2} + 2\pi l\right)}}\right) \\
&= \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-j(\pi + 4\pi l)}}\right) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-j(\pi + 4\pi l)}}\right) \\
&= \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-j\pi}e^{-j4\pi l}}\right) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi/2 - 2\pi l)\left(\frac{4}{4 + e^{-j\pi}e^{-j4\pi l}}\right)
\end{aligned}$$

since $e^{-j4\pi l} = 1$ for all values of l and $e^{j\pi} = -1$

$$\begin{aligned}
&= \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi/2 - 2\pi l)\left(\frac{4}{(-1)(1)}\right) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi/2 - 2\pi l)\left(\frac{4}{4 + (-1)(1)}\right) \\
&= \frac{4}{3}\left(\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi/2 - 2\pi l) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi/2 - 2\pi l)\right)
\end{aligned}$$

Now apply the inverse Fourier transform (using the same basic discrete-time Fourier transform pair):

$$\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)) \stackrel{\mathrm{FT}}{\longleftrightarrow} \cos \omega_0 n$$

Then signal $y[n]$ is:

$$\boxed{y[n] = \frac{4}{3}\cos\left(\frac{\pi n}{2}\right)}$$

c) Fourier transform pair is (we're using convolution property of discrete time Fourier transform):

$$y[n] = x[n] * h[n] \stackrel{\mathrm{FT}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Then $Y(e^{j\omega})$ is:

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \left(3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega}\right)\left(-e^{j\omega} + 2e^{-j2\omega} + e^{j4\omega}\right) \\
&= 3e^{j5\omega} + e^{j4\omega} - e^{j3\omega} - 3e^{j2\omega} + e^{j\omega} + 1 + 6e^{-j\omega} - 2e^{-j3\omega} + 4e^{-j5\omega}
\end{aligned}$$

Now apply the inverse Fourier transform using the basic discrete-time Fourier transform pair:

$$e^{-j\omega n_0} \stackrel{\mathrm{FT}}{\longleftrightarrow} \delta[n - n_0]$$

$$\boxed{y[n] = 3\delta[n+5] + \delta[n+4] - \delta[n+3] - 3\delta[n+2] + \delta[n+1] + \delta[n] + 6\delta[n-1] - 2\delta[n-3] + 4\delta[n-5]}$$