Signals and Systems

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Chapter 2
Linear Time-Invariant System

2.0 Introduction

Two reasons we focus on the properties, ***linearity*** and ***time invariance***:

- First, many physical processes possess these properties and thus can be modeled as ***linear time-invariant*** (LTI) systems.
- In addition, LTI systems can be analyzed in considerable detail, providing both insight into their properties and a set of powerful tools that form the core of signal and system analysis.

- If we can represent the ***input*** to an LTI system in terms of ***a linear combination of a set of basic signals***, we can then use superposition to compute the output of the system in terms of ***its response to the basic signals***.
- One of the important characteristic of the ***unit impulse***, in both the discrete-time (D-T) and continuous-time (C-T), is that very general signals can be represented as ***linear combination of delayed impulses***.

- This fact, together with the properties of ***superposition*** and ***time invariance***, is used to develop a complete characterization of any LTI system in terms of its ***response to a unit impulse***.
  - D-T case: convolution ***sum***
  - C-T case: convolution ***integral***

2.1 Discrete-time LTI system: The convolution sum

2.1.1 The representation of discrete-time signals in terms of impulses

The key idea in visualizing how the discrete-time unit impulse can be used to construct any discrete-time signals is to think of a discrete-time signal as a sequence of ***individual impulse***.

Consider the signal $x[n]$ depicted in Figure 2.1(a). We have

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \qquad \text{(Eq. 2.2)}$$

To consider the unit step signal $u(t)$, we can construct it using the unit impulse signals as:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

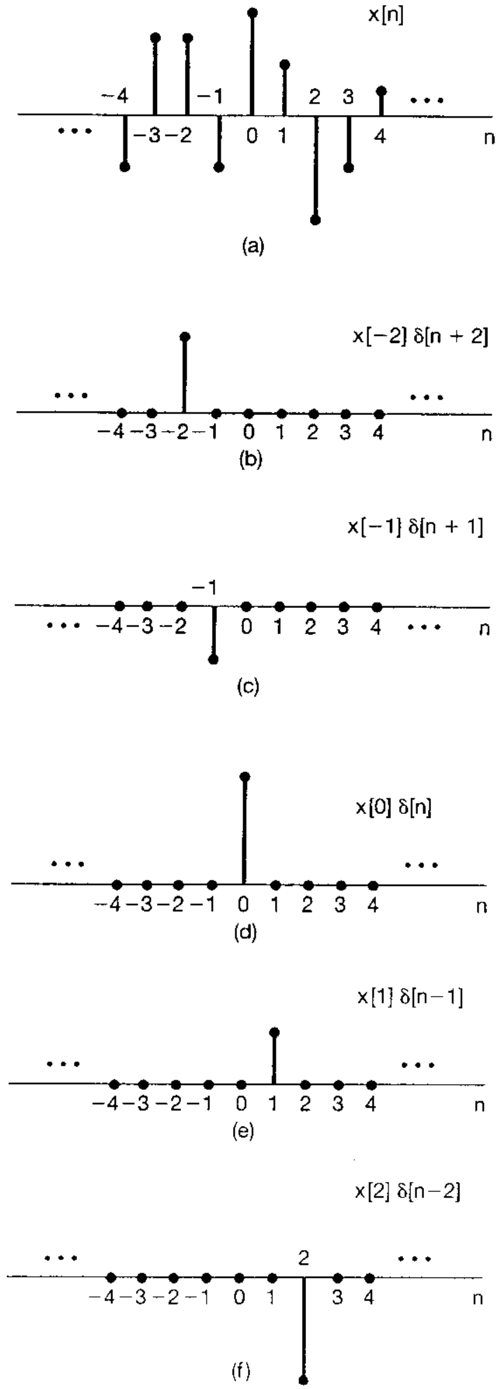


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

- $x[n] = \ldots\, x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\, \delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots$

- An arbitrary sequence can be represented as a ***linear combination*** of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are $x[k]$.
- For example, $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

2.1.2 The discrete-time unit impulse response and the convolution-sum representation of LTI systems

With the arbitrary input $x[n]$ to a linear (but possibly time-varying) system expressed in the form of the eq. (2.2), let $\boldsymbol{h_k[n]}$ denote the ***response*** of the linear system to the shifted unit impulse $\boldsymbol{\delta[n-k]}$, the ***output*** $y[n]$ can be expressed as

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]. \qquad (2.3)$$

- According to eq. (2.3), if we know the ***response*** of ***a linear system to the set of shifted unit impulses***, we can construct the response to an arbitrary input.
- If $h_k[n]$ is ***known***, the response to an arbitrary input can be constructed.
- The system output $y[n]$ at time n is simply the ***superposition*** of the response due to the input value at each point in time.

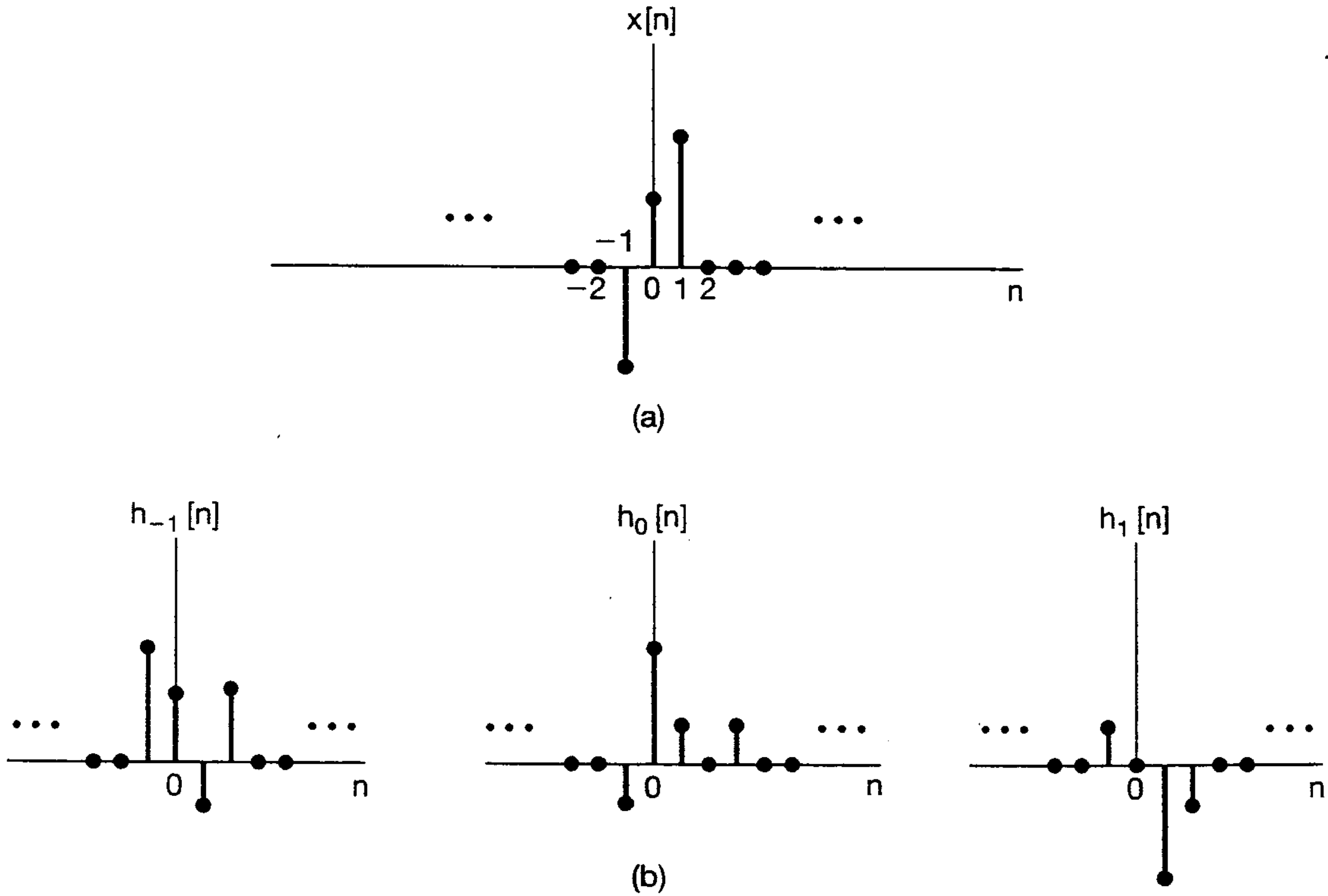


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

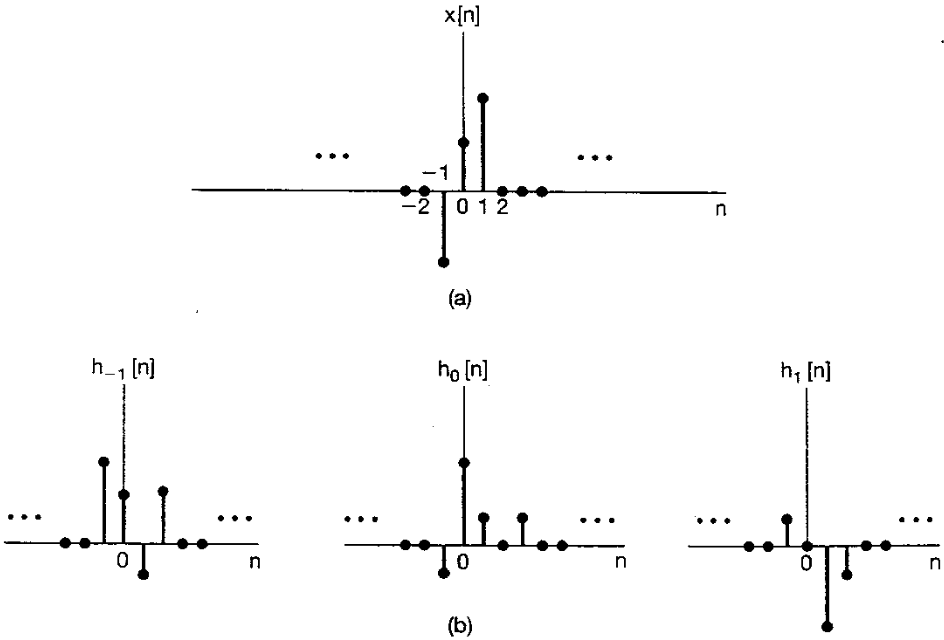
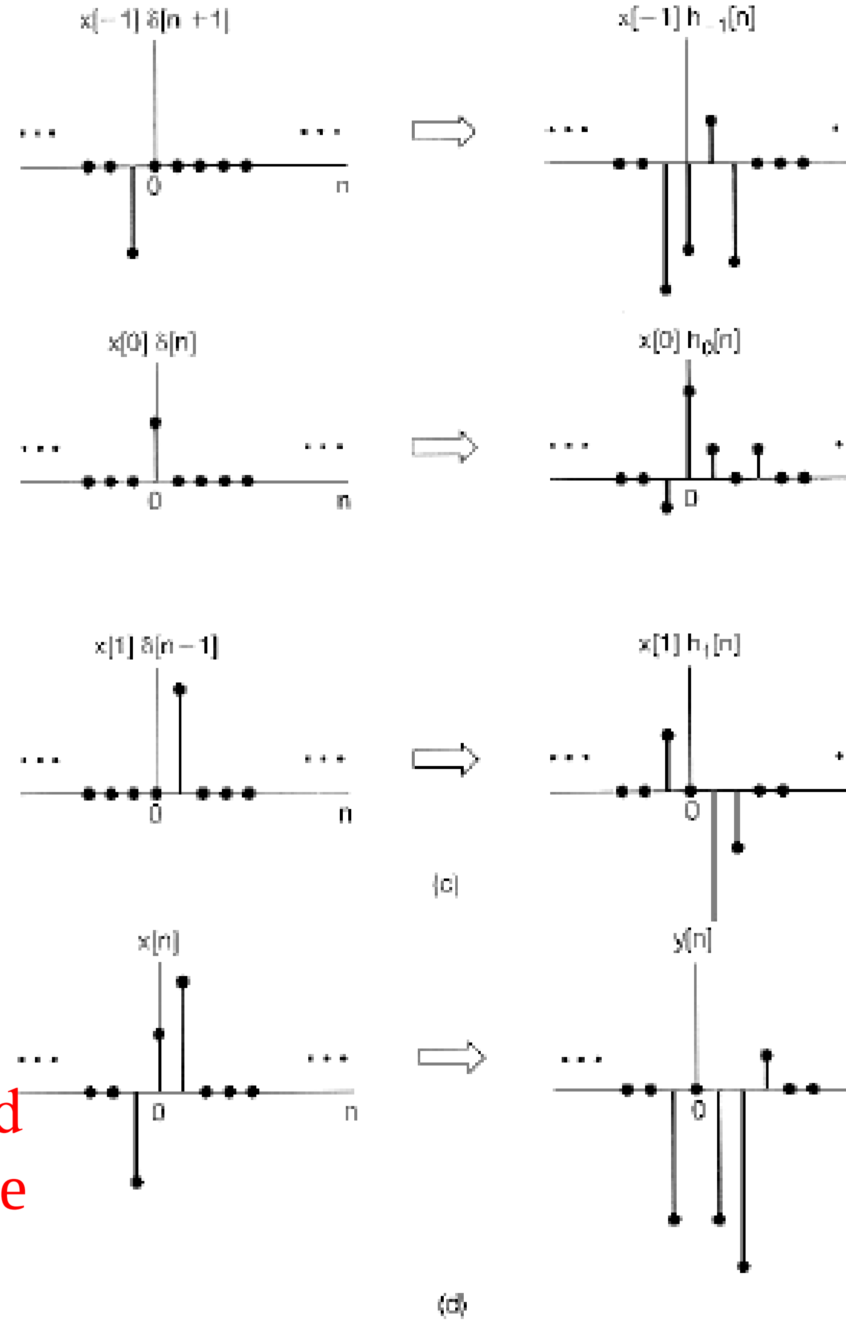


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).



$$y[n] =$$

$$\ldots + x[-1]h_{-1}[n] + x[0]h_0[n]$$

$$+ x[1]h_1[n] + \ldots$$

Here $x[-1]$, $x[0]$, $x[1]$,… are constants and thus are multiplied to each non-zero value of $h_k[n]$

If the linear system is also ***time invariant***, then these ***responses to time-shifted unit impulses are all time-shifted versions of each other***. Specifically, since $\delta[n]$, the response $h_k[n]$ is a time shifted version of $h_o[n]$;i.e.,

$$h_k[n]=h_0[n-k]. \tag{2.4}$$

For notational convenience, let

$$h[n]=h_o[n]. \tag{2.5}$$

Then eq.(2.3) becomes

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]. \tag{2.6}$$

This result is referred to as the ***convolution sum*** or ***superposition sum***. We will represent the operation of convolution symbolically as

$$y[n]=x[n]*h[n] \tag{2.7}$$

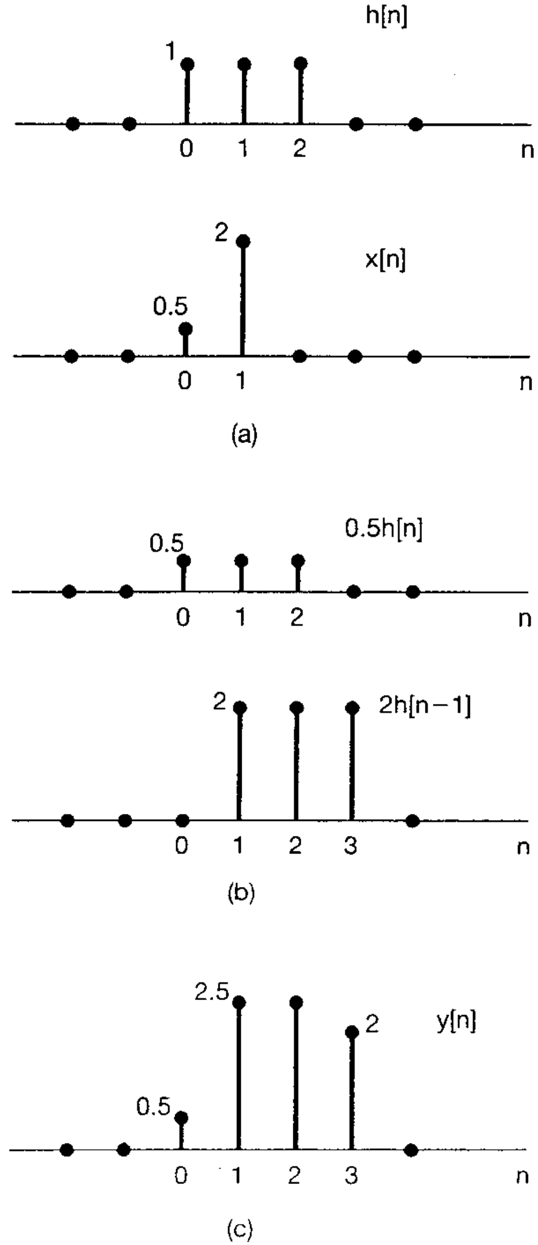


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

- The calculation of convolution can be displayed graphically. It begins with the two signals $x[k]$ and $h[n-k]$ and is a function of k.
  - How to get $x[k]$ and $h[n-k]$?
  - $x[n] \rightarrow x[k]$, $h[n] \rightarrow h[k] \rightarrow h[-k] \rightarrow h[n-k]$
- $g[k]=x[k]h[n-k]$, at each time k, it represents the contribution of $x[k]$ to the output at time n.
- Summing all $g[k]$ at each k yields the output value $y[n]$ at the selected time n.

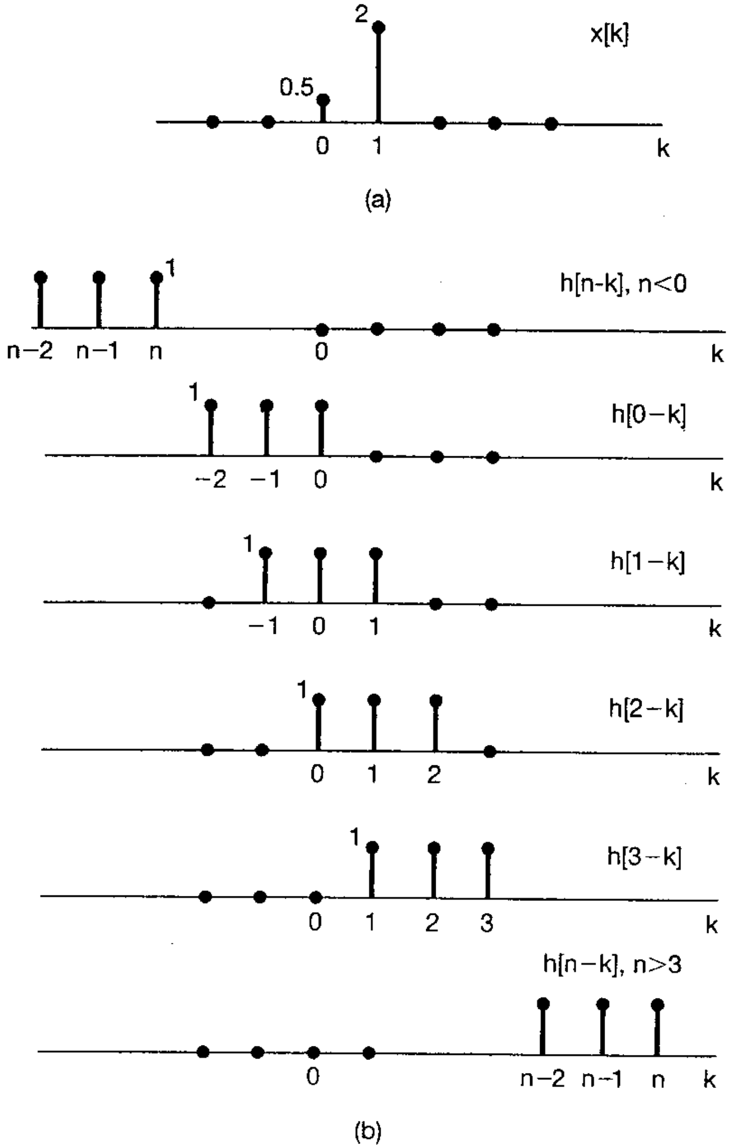


Figure 2.4 Interpretation of eq. (2.6) for the signals $h[n]$ and $x[n]$ in Figure 2.3; (a) the signal $x[k]$ and (b) the signal $h[n-k]$ (as a function of k with n fixed) for several values of n ($n < 0$; $n = 0, 1, 2, 3$; $n > 3$). Each of these signals is obtained by reflection and shifting of the unit impulse response $h[k]$. The response $y[n]$ for each value of n is obtained by multiplying the signals $x[k]$ and $h[n-k]$ in (b) and (c) and then summing the products over all values of k. The calculation for this example is carried out in detail in Example 2.2.

Example 2.2: n determines the position of $h[n\text{-}k]$

Example 2.2

For n<0, x[k]h[n-k]=0 for all k

y[n]=0 for n <0

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5, \qquad y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 2.5,$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 2.5, \qquad y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0$$

for n>3, x[k]h[n-k]=0 for all k

Example 2.3

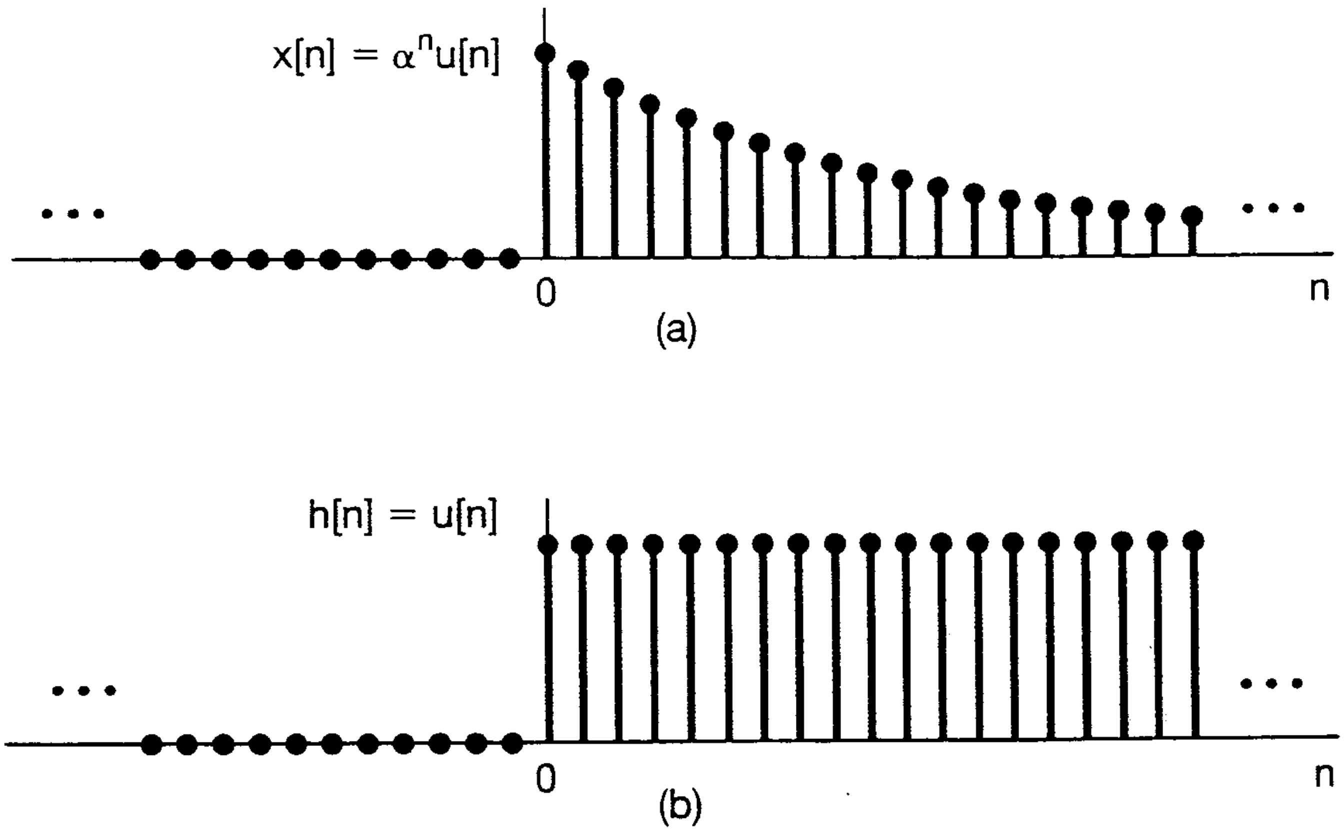


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

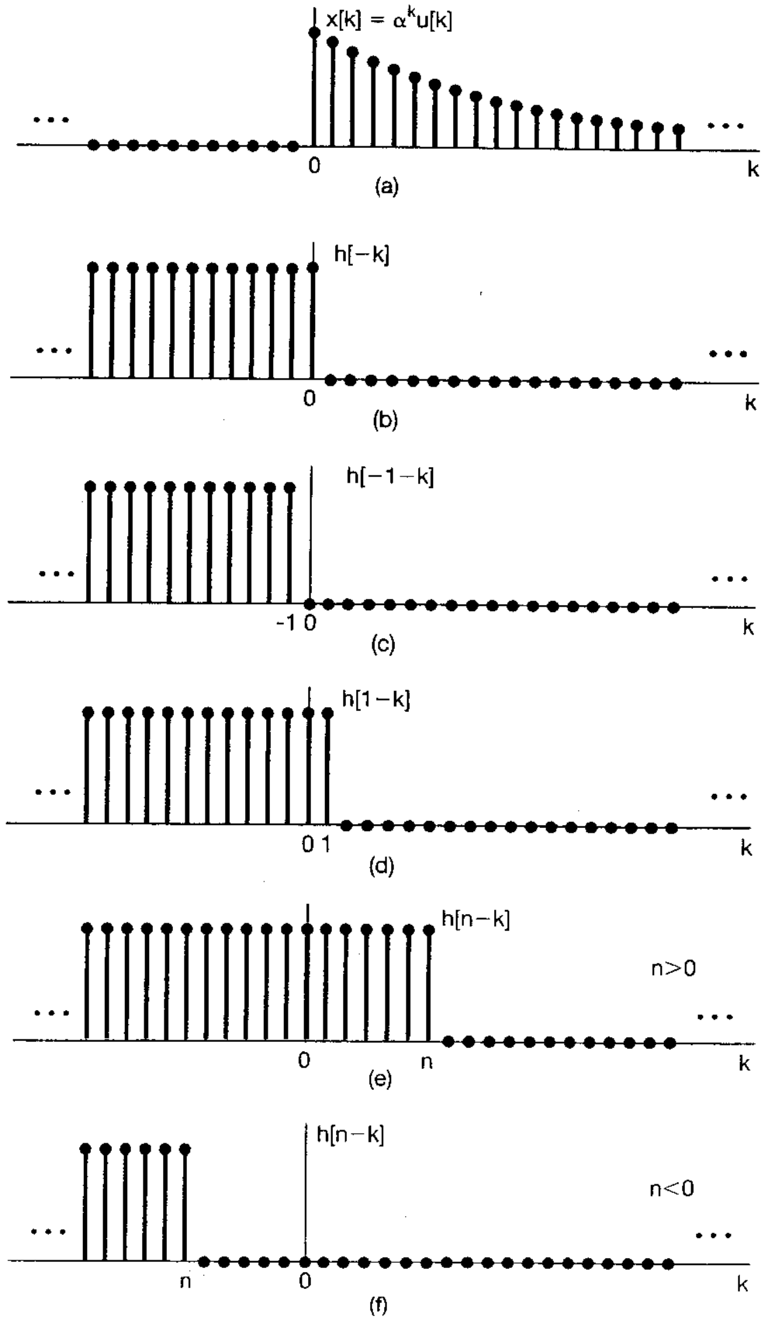


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

$n < 0,\ x[k]h[n-k] = 0 \text{ for all } k,$

$\therefore y[n] = 0\ \ \forall n < 0$

$$n \ge 0,\ x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha},\ \text{for } n \ge 0$$

$$= \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n],\ \text{see Fig. 2.7}$$

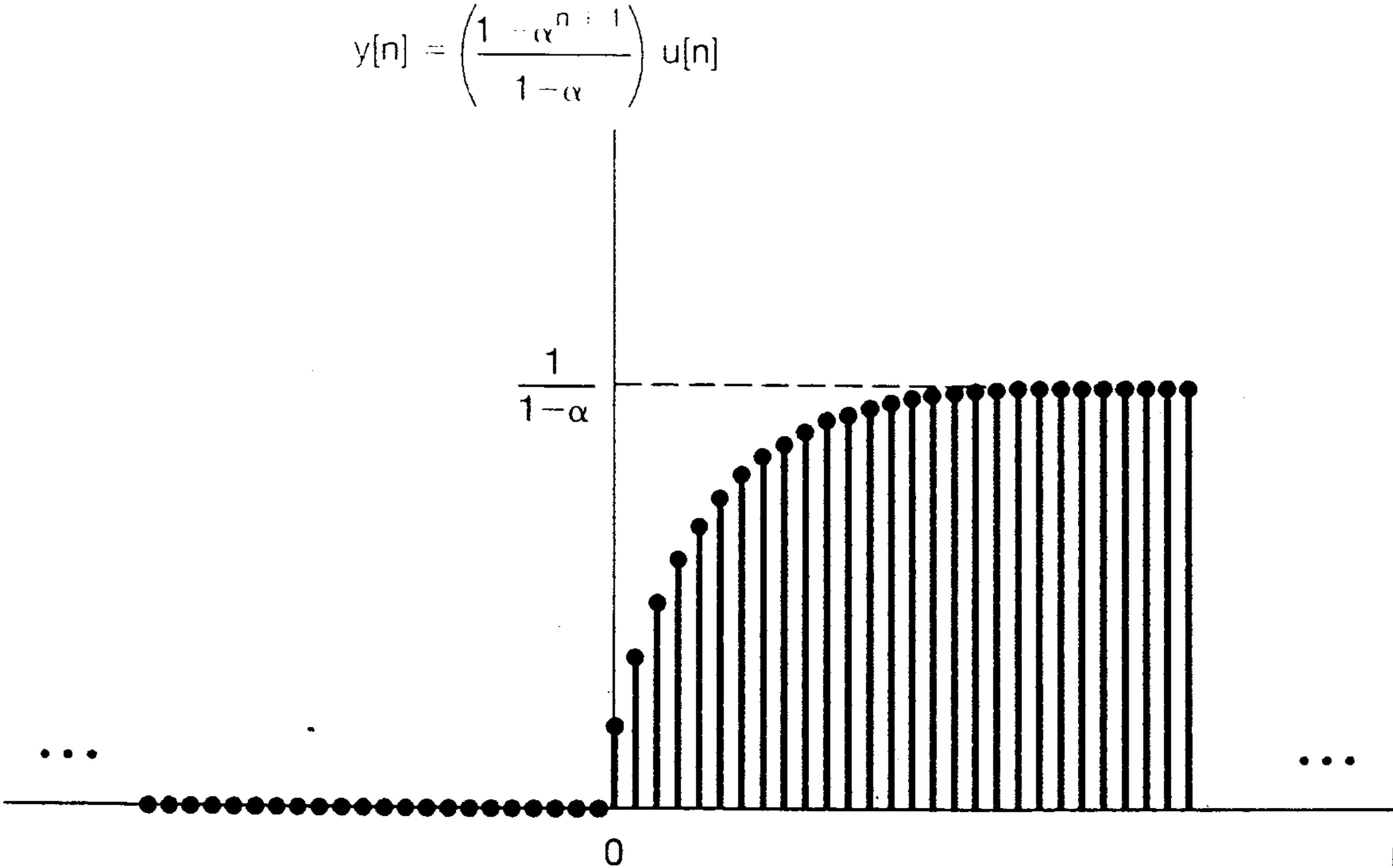


Figure 2.7 Output for Example 2.3.

Example 2.4

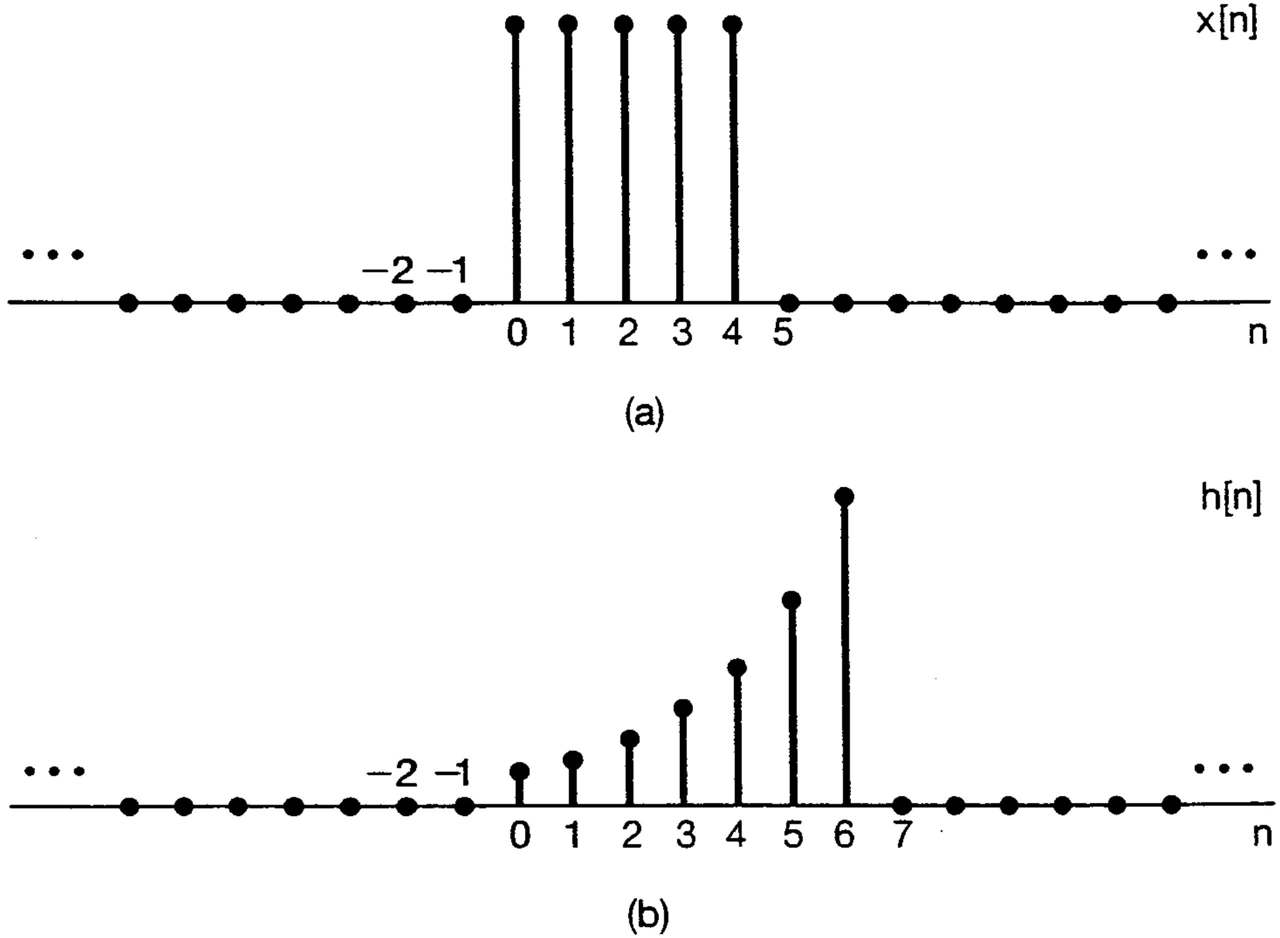


Figure 2.8 The signals to be convolved in Example 2.4.

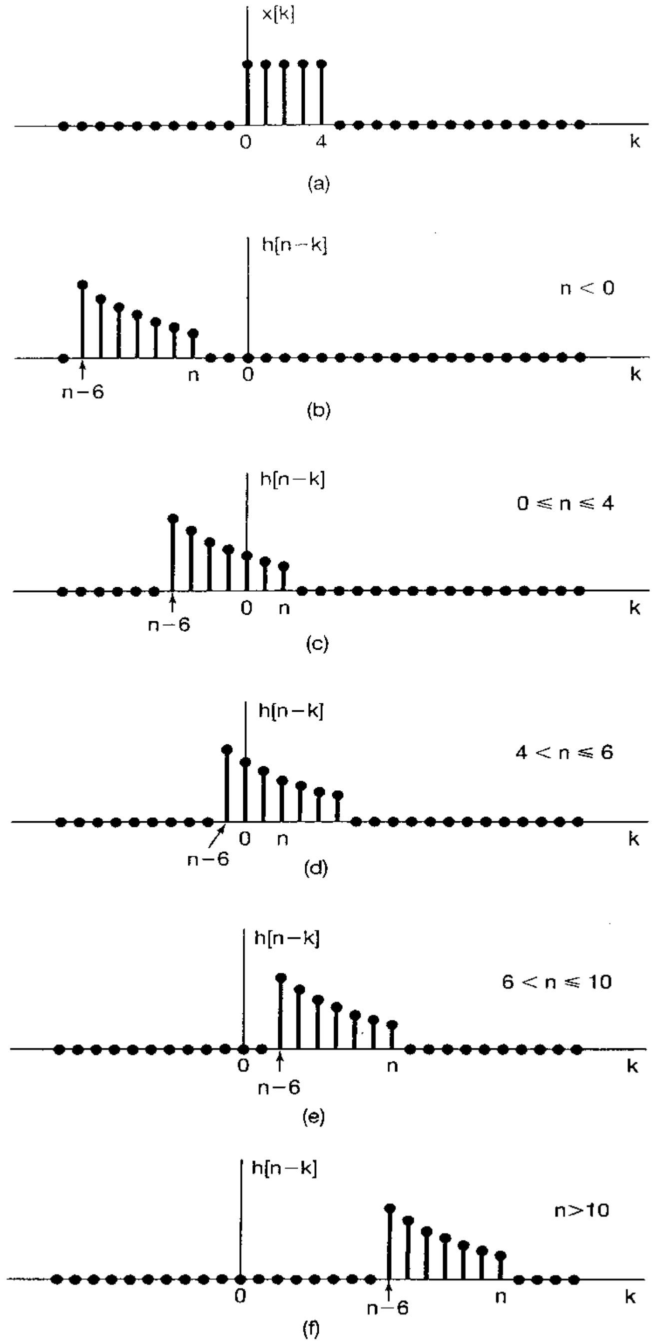


Figure 2.9 Graphical interpretation of the convolution performed in Example 2.4.

- $x[n] = \begin{cases} 1, 0 \leq n \leq 4 \\ 0, \text{ otherwise} \end{cases},$

- $\mathrm{h}[n] = \begin{cases} \alpha^n, 0 \leq n \leq 6 \\ 0, \text{ otherwise} \end{cases}$

→for n<0, no overlap between $x[k]$ and $h[n\text{-}k]$

→for 0<=n<=4,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq n \\ 0\,, & \text{otherwise} \end{cases}$$

→for 4<n<=6,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

→for 6<n<=10,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & n-6 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

→for n>10, no overlap between $x[k]$ and $h[n\text{-}k]$

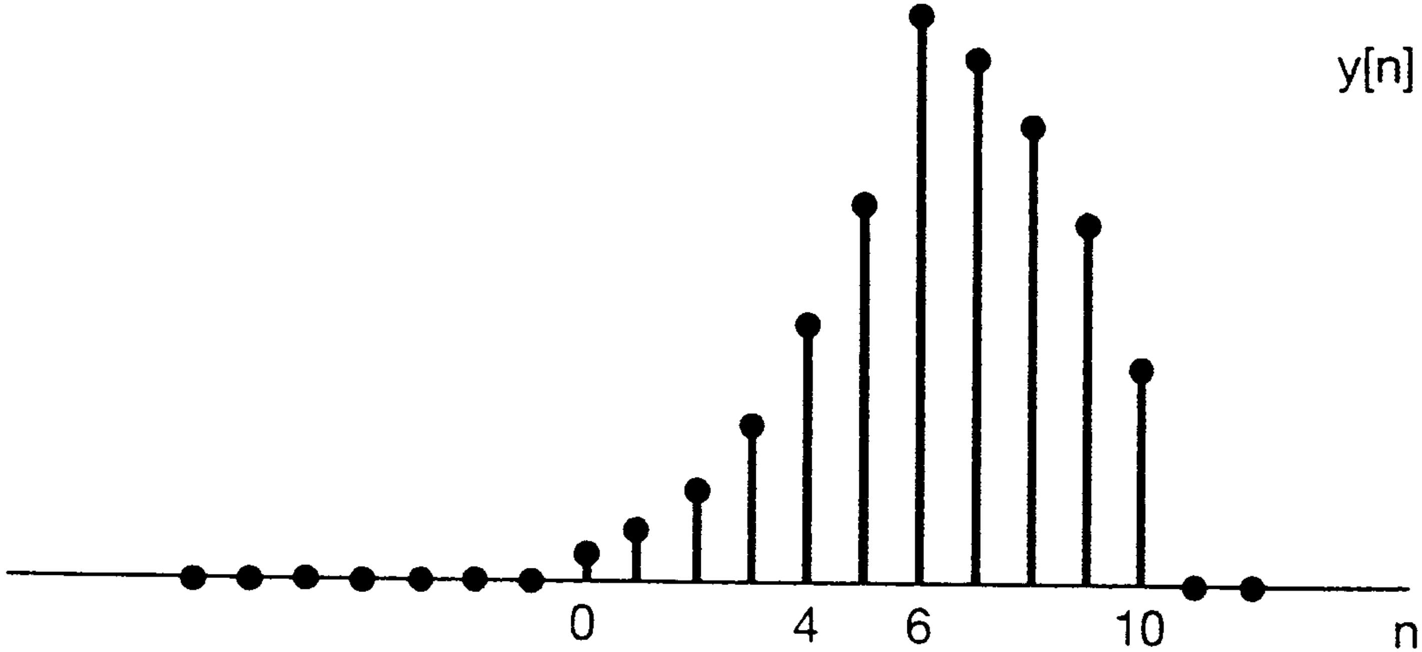


Figure 2.10 Result of performing the convolution in Example 2.4.

Example 2.5

Example 2.5

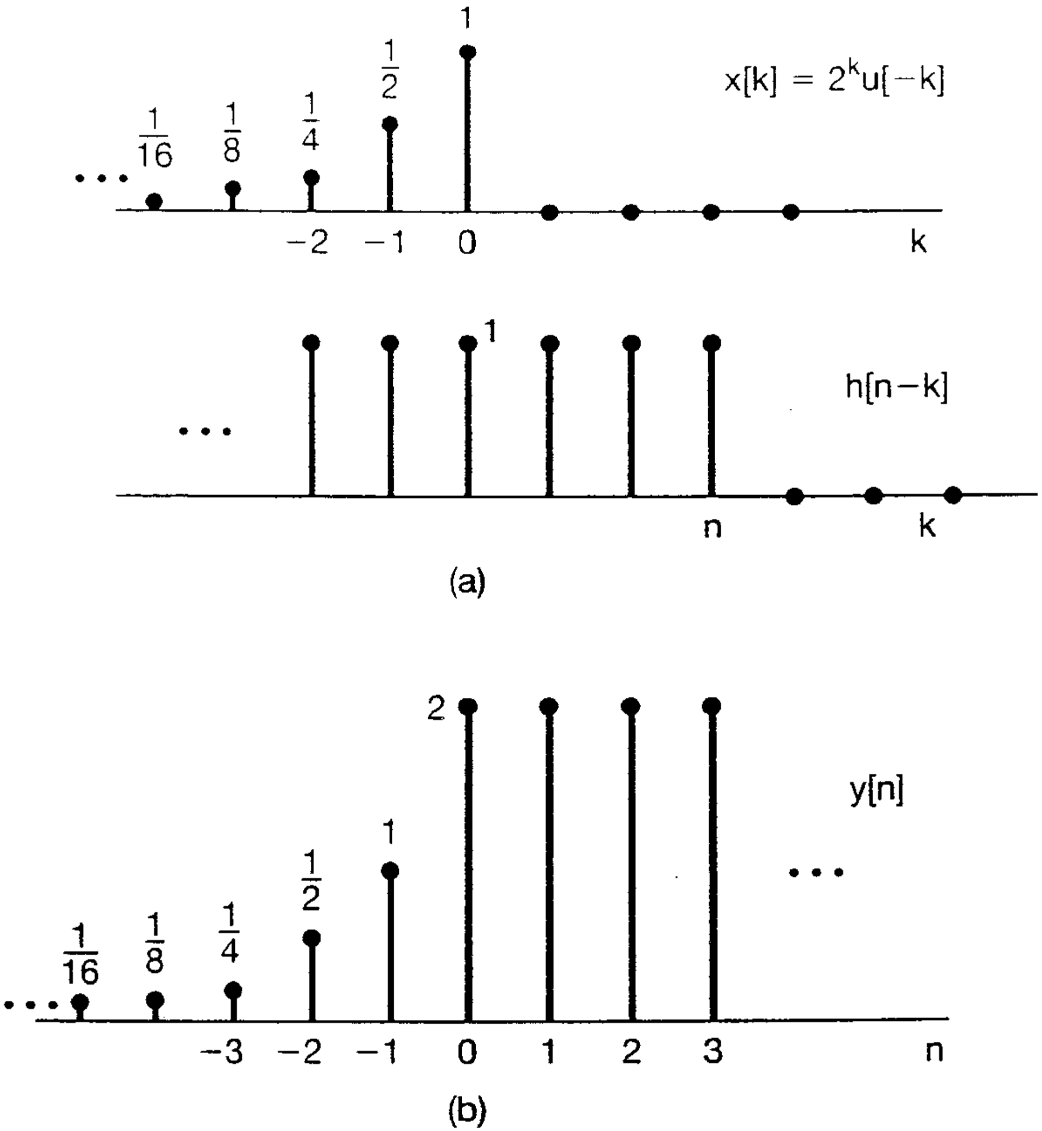


Figure 2.11 (a) The sequences $x[k]$ and $h[n-k]$ for the convolution problem considered in Example 2.5; (b) the resulting output signal $y[n]$.

Example 2.5

- $x[n] = 2^n u[-n], h[n] = u[n]$ → $x[k]$=0 for k>0, $h[n$-$k]$=0 for k>n

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[0-k] = \sum_{k=-\infty}^{0} 2^k \quad \text{(for n>=0)}$$

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \qquad 0 < |\alpha| < 1 \;\; (\textit{\textbf{infinite sum formula}})$$

$$\rightarrow \sum_{k=-\infty}^{0} 2^k = \sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-\frac{1}{2}} = 2 \;\; (n > 0)$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k \;\; (\text{n} < 0)$$

$$= \sum_{l=-n}^{\infty} (\frac{1}{2})^l = \sum_{m=0}^{\infty} (\frac{1}{2})^{m-n} = (\frac{1}{2})^{-n} \sum_{m=0}^{\infty} (\frac{1}{2})^m = 2^n \cdot 2 = 2^{n+1}$$

2.2 Continuous-time LTI Systems: The Convolution Integral

- 2.2.1 The representation of continuous-time signals in terms of impulses
  - In the preceding section, we can think of the discrete-time system as responding to a sequence of individual impulse. In continuous time, we ***do not*** have a discrete sequence of input values. If we think of the unit impulse as the idealization of a pulse which is so short that its duration is inconsequential for any real, physical system, we can develop a representation for arbitrary continuous-time signals in terms of these ***idealized pulses*** with ***vanishingly small duration***, or equivalently, ***impulses***.

2.2.1 The representation of continuous-time signals in terms of impulses

To develop the continuous-time counterpart of the discrete-time shifting property in eq. (2.2), we begin by considering a pulse or "***staircase***" approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$, as illustrated in Figure 2.12(a).

If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta, \\ 0, & \text{Otherwise.} \end{cases}$$

Then, we have the expression

$$\hat{y}(t)(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \quad (2.25)$$

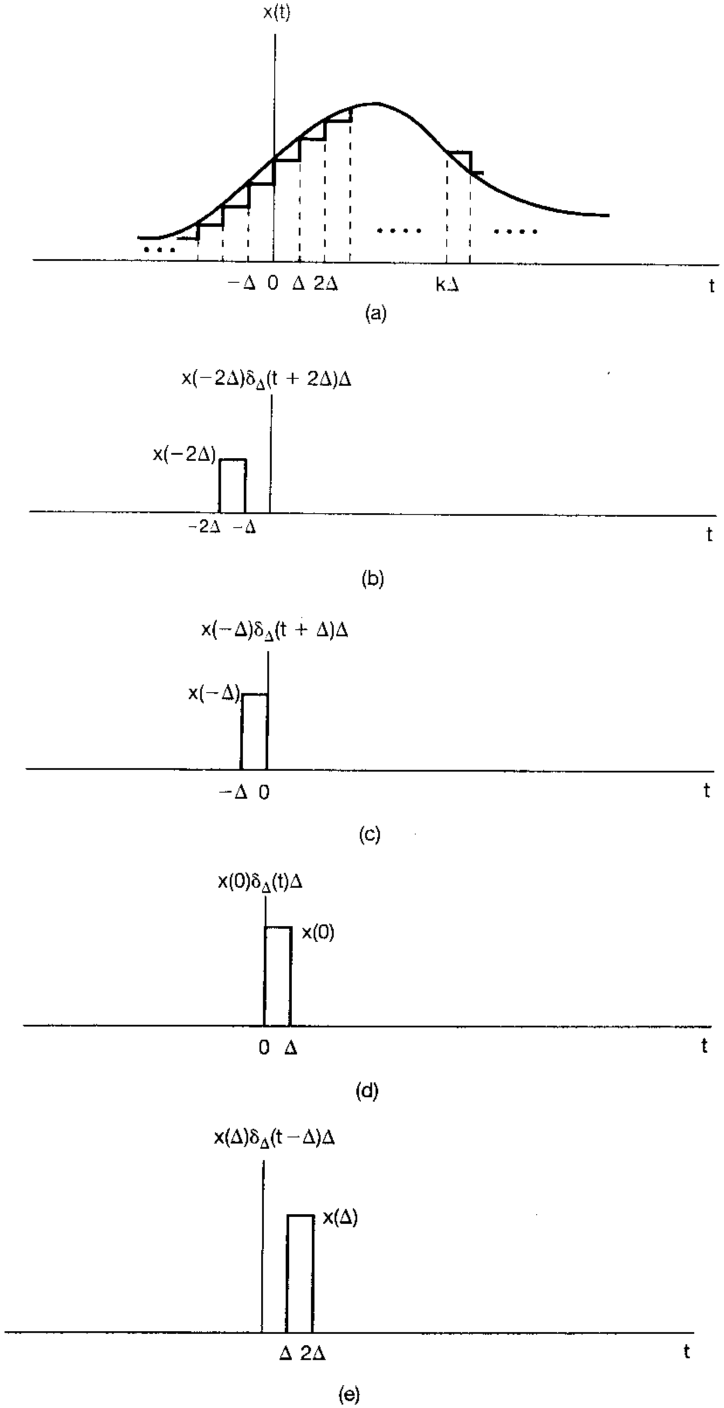


Figure 2.12 Staircase approximation to a continuous-time signal.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$.

Therefore,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta . \qquad (2.26)$$

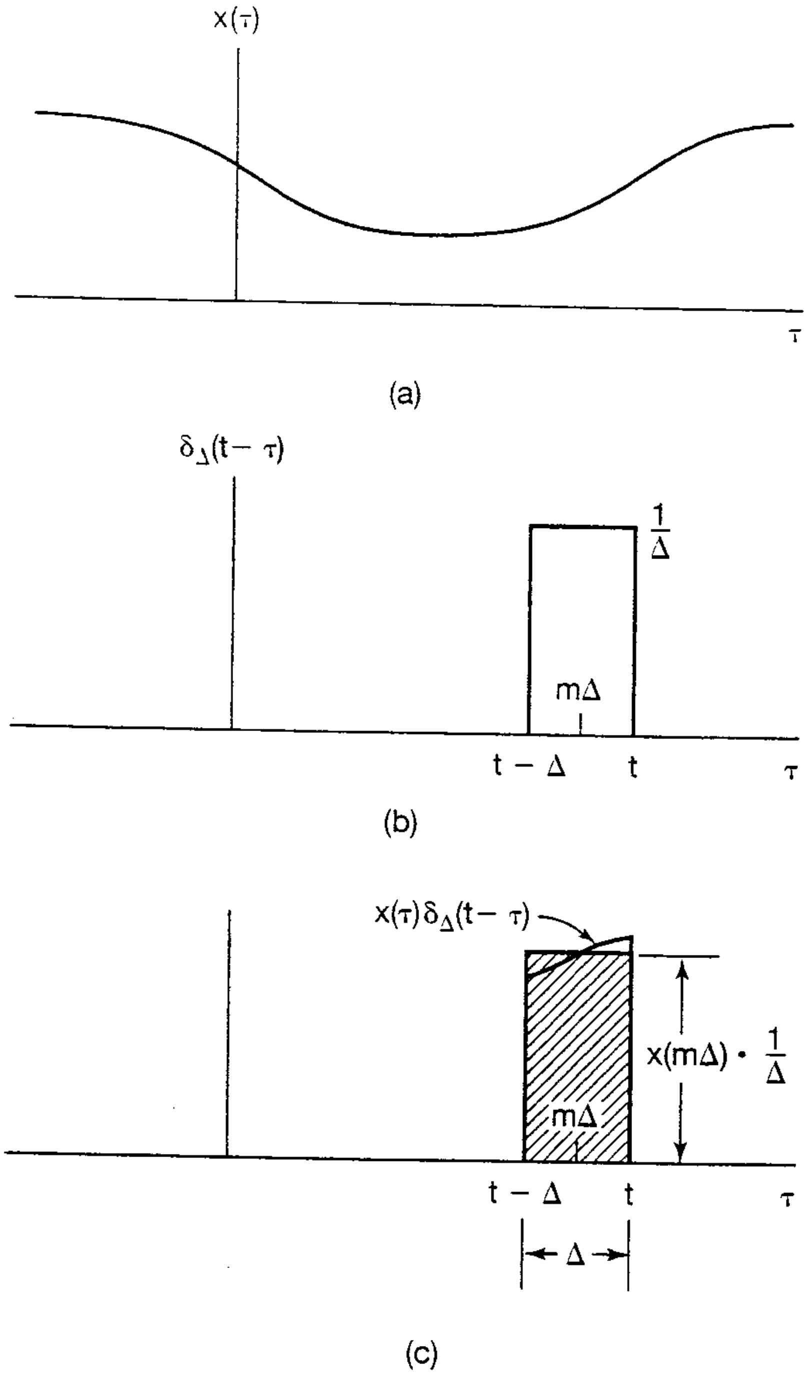
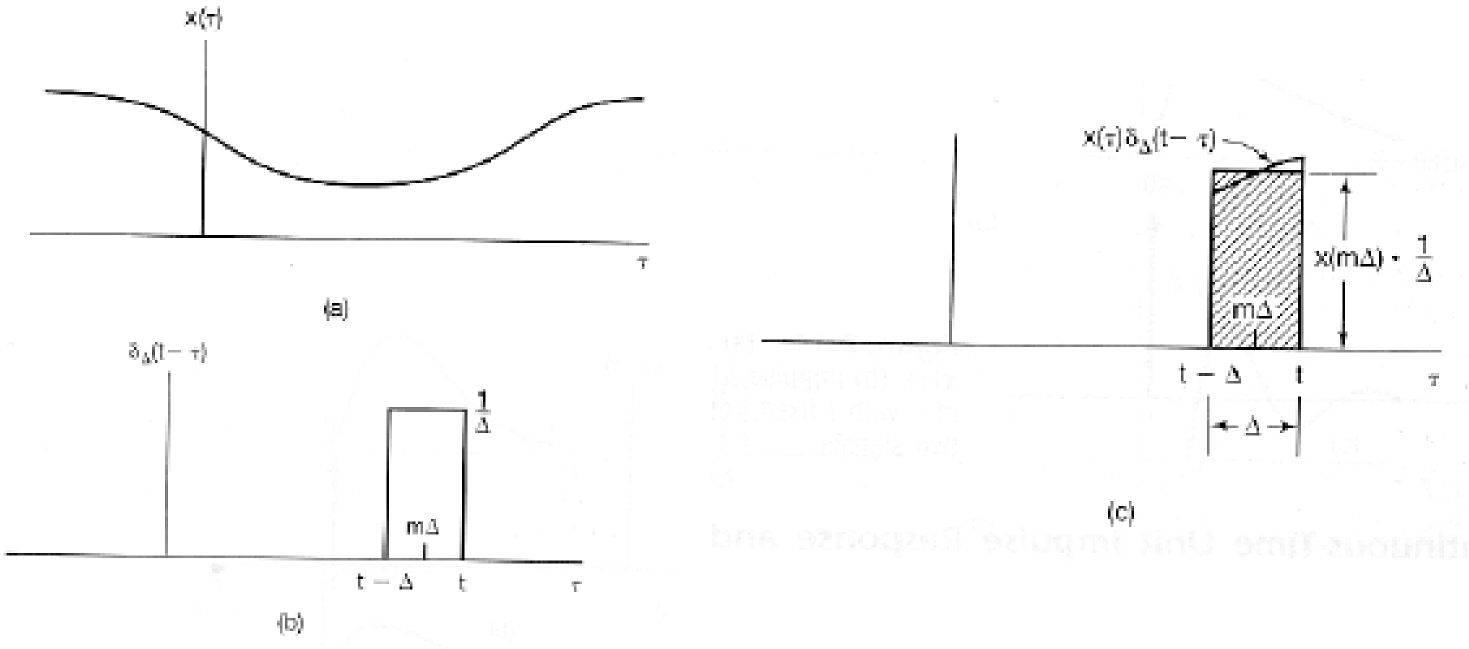


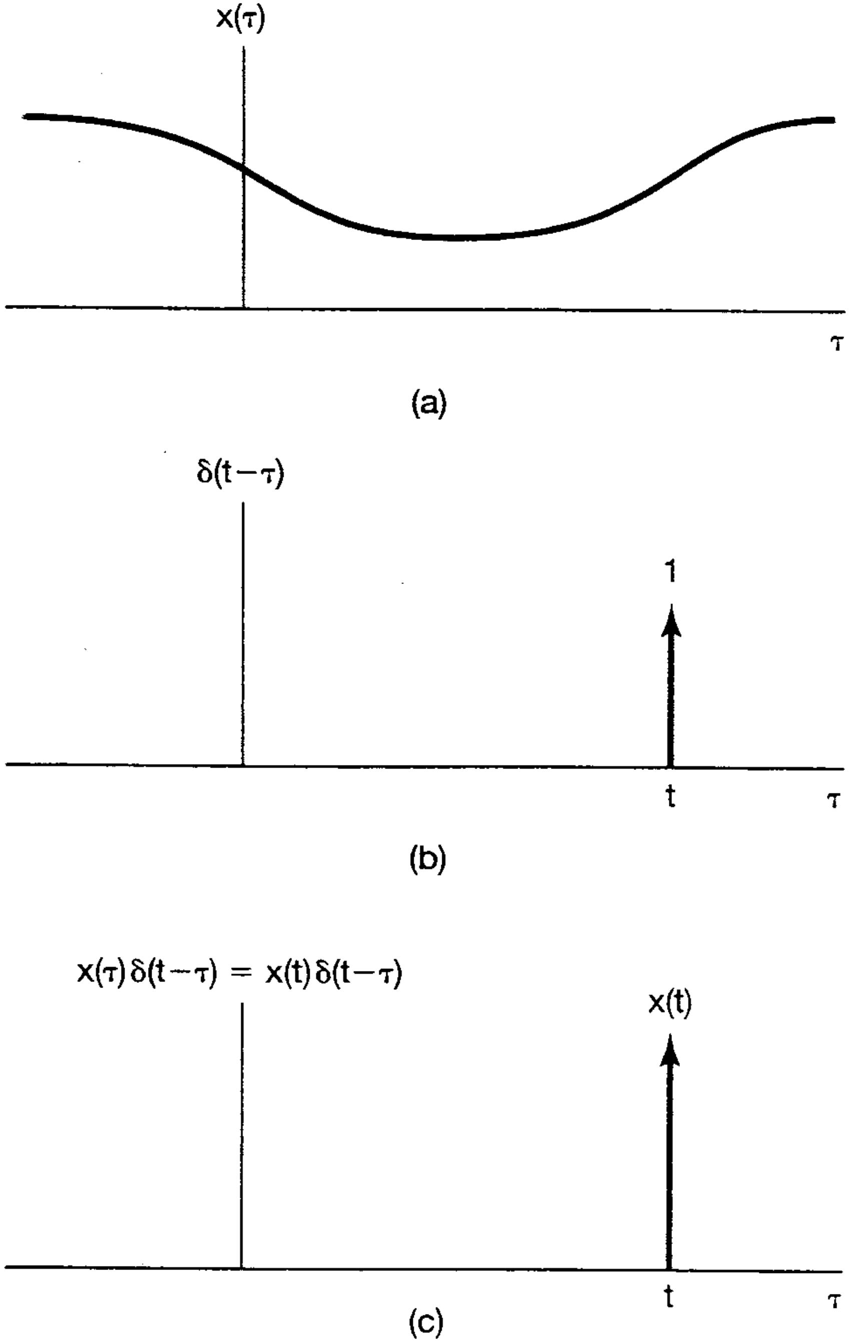
Figure 2.13 Graphical interpretation of eq. (2.26).

Also, as $\Delta \rightarrow 0$, the summation in eq. (2.26) approach an integral. Illustrated in figure 2.13.



consequently, $$x(t)=\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau. \qquad (2.27)$$

Signal $x(t)$ can be represented as a sum (more precisely, an integral) of weighted, shifted impulses, as shown in Eq. (2.27)



$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$

Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a functio of τ with t fixed; (c) product of these two signals.

2.2.2 The continuous-time unit impulse response and the convolution integral representation of LTI systems

Eq. (2.25) represents the signal $\hat{x}(t)$ as a sum of ***scaled*** and ***shifted*** versions of the basic pulse signal $\delta_\Delta(t)$.

Let us define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$.

The response $\hat{y}(t)$ of a linear system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$. Then

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta\,. \qquad (2.29)$$

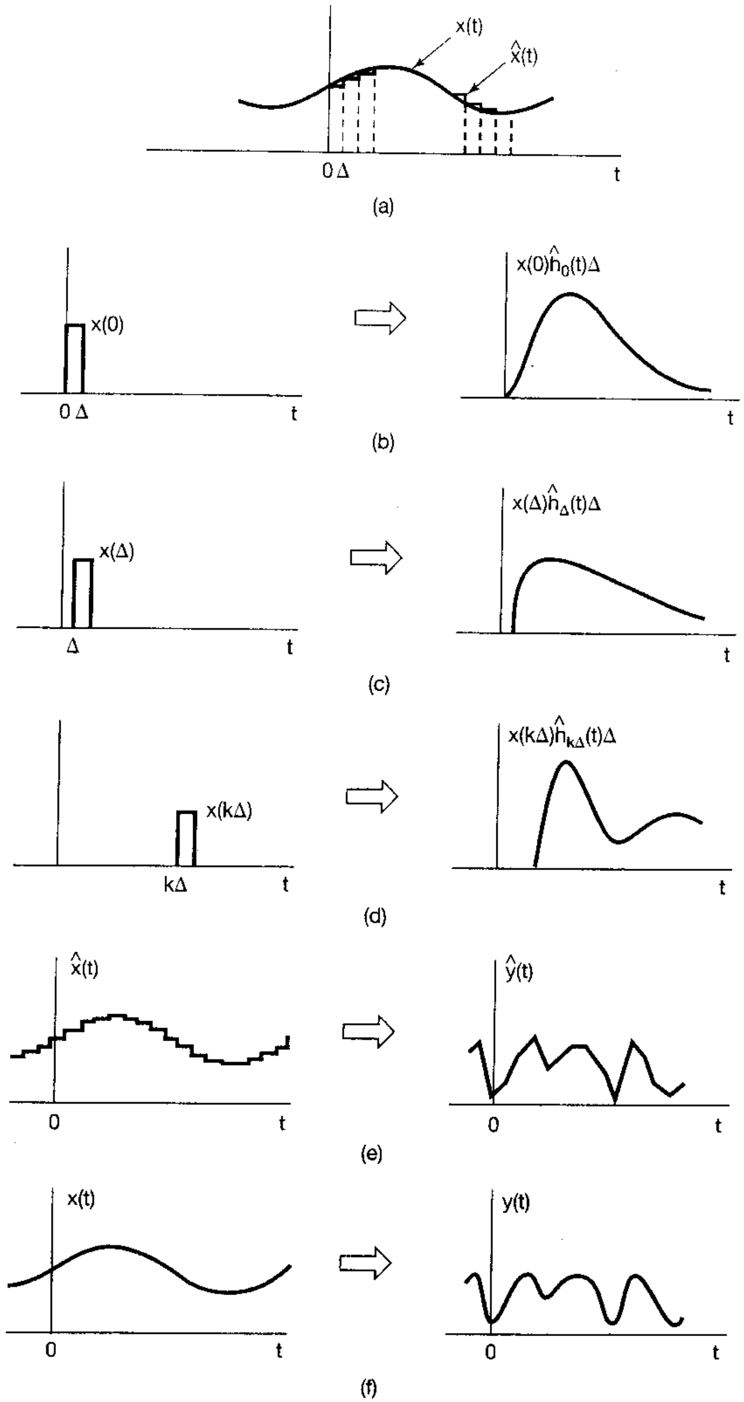


Figure 2.15 Graphical interpretation of the response of a continuous-time linear system as expressed in eqs. (2.29) and (2.30).

Since the pulse $\delta_\Delta(t-k\Delta)$ corresponds to a shifted unit impulse as $\Delta\to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit. Therefore, if we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ, then

$$y(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty} x(k\Delta)h_{k\Delta}(t)\Delta\,. \qquad (2.30)$$

As $\Delta\to 0$, the equation can seen graphically in Fig. 2.16.

Therefore,

$$y(t)=\int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau\,. \qquad (2.31)$$

and input $\quad x(t)=\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau\,.$

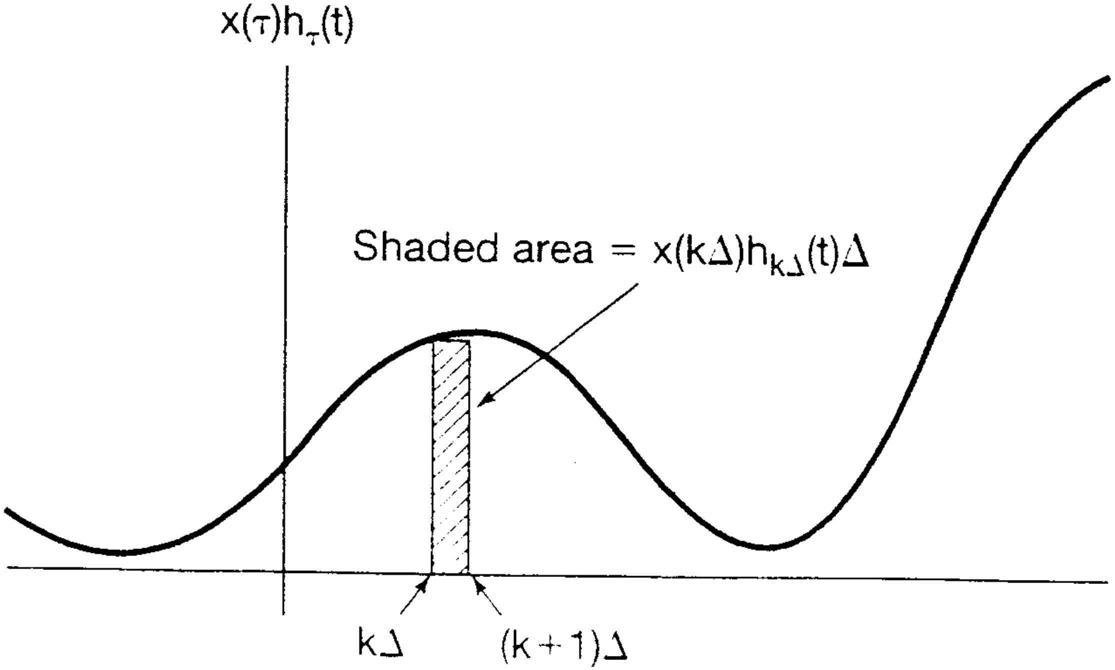


Figure 2.16 Graphical illustration of eqs. (2.30) and (2.31).

- If the system is also ***time-invariant***, then

$$h_\tau(t) = h(t-\tau)$$

As the same conception of discrete-time system, eq. (2.31) can seen as

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\,. \qquad (2.33)$$

This result is referred to as the ***convolution integral*** or superposition integral . We will represent the operation of convolution symbolically as

$$y(t) = x(t) * h(t). \qquad (2.34)$$

- A continuous-time LTI system is completely characterized by its ***impulse response h(t)***.
- The procedure for evaluating the convolution integral:
  - $h(\tau) \rightarrow h(t-\tau)$: Reflecting about the origin and shift to the right by t if $t > 0$, or a shift to the left by $|t|$ for $t < 0$
  - $x(\tau) \times h(t-\tau)$: $y(t)$ is obtained by integrating the resulting product from $\tau = -\infty$ to $\tau = \infty$

Example 2.6

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$h(t) = u(t)$$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau} \Big|_0^t$$

$$= -\frac{1}{a} e^{-at} - (-\frac{1}{a})$$

$$= \frac{1}{a}(1 - e^{-at}), \quad t > 0$$

$$= \frac{1}{a}(1 - e^{-at})u(t)$$

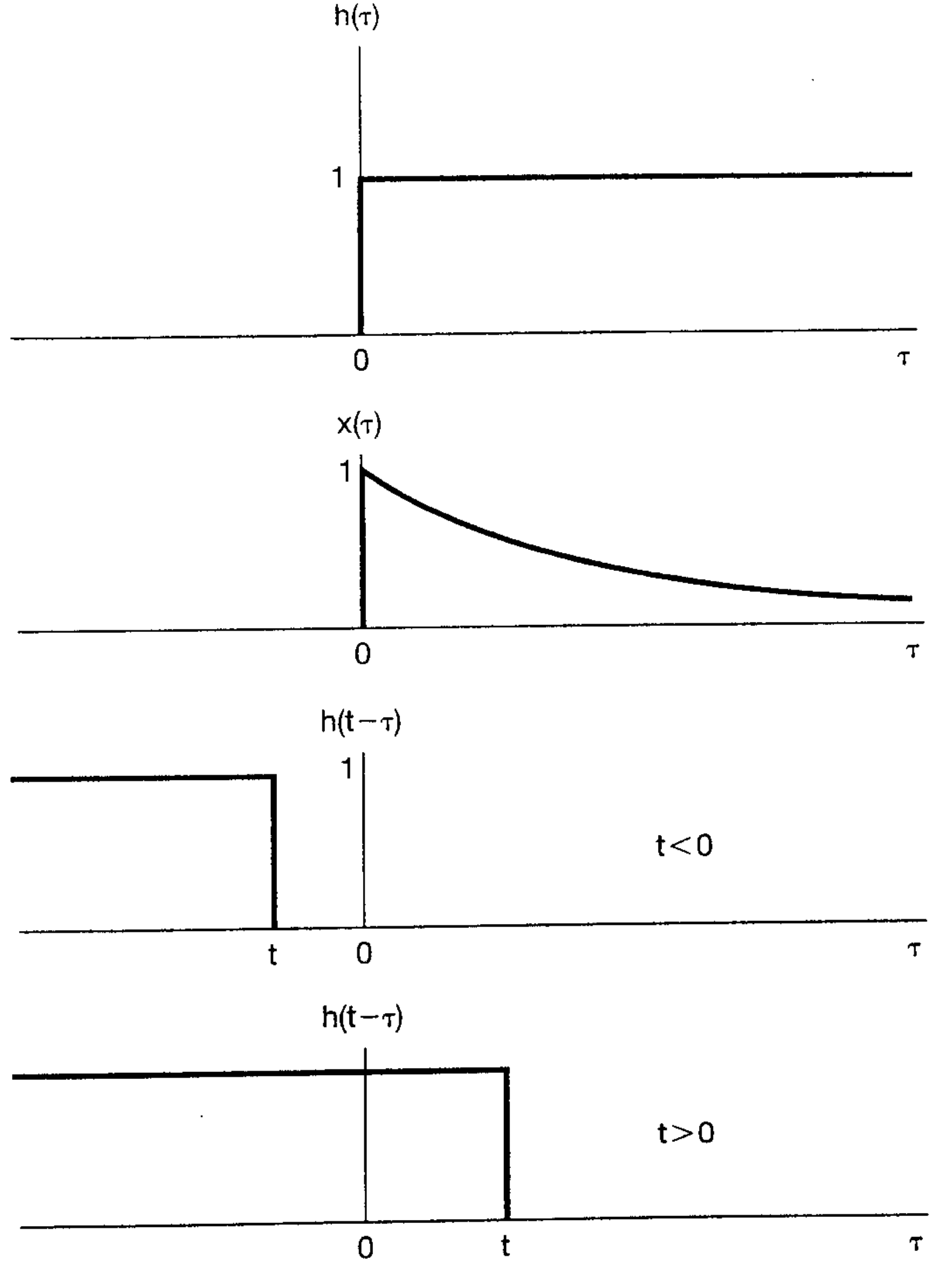


Figure 2.17 Calculation of the convolution integral for Example 2.6.

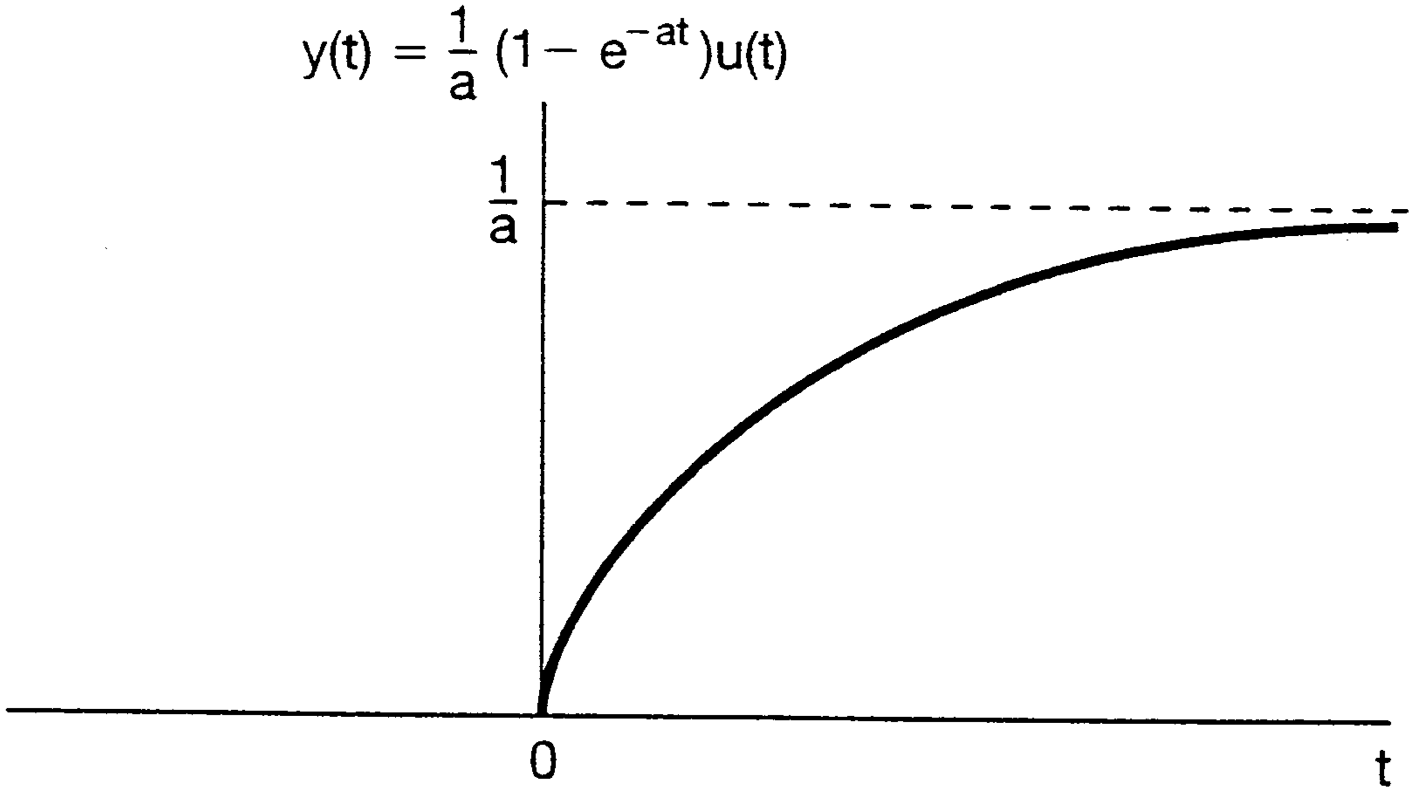


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Example 2.7

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

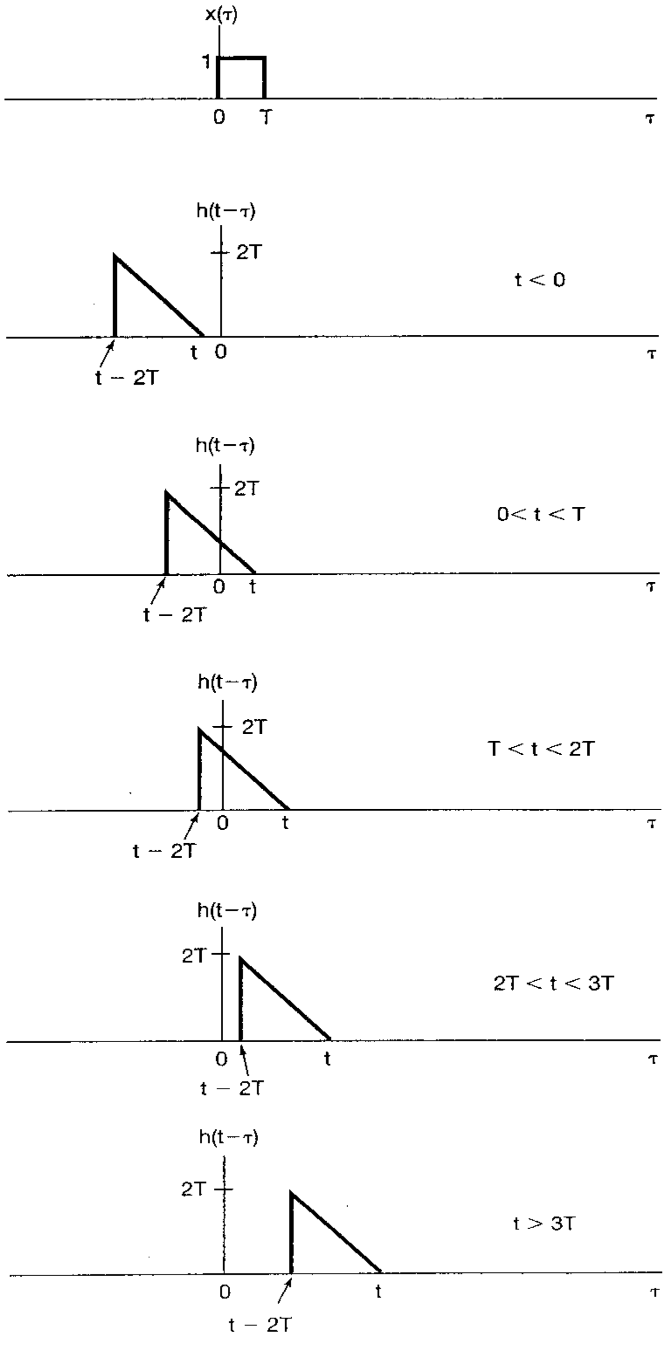


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

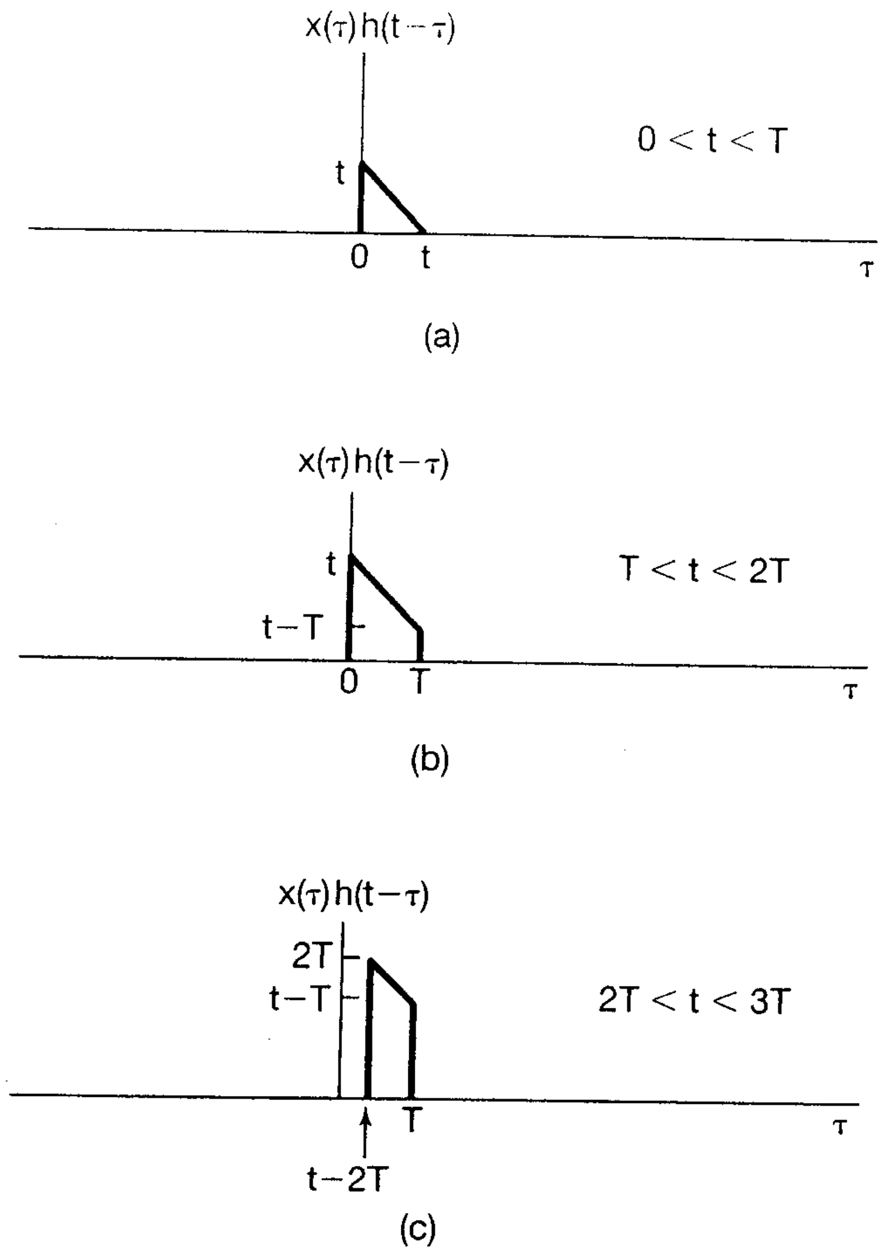


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

Example 2.7

$$x(t) = \begin{cases} 1, 0 < t < T \\ 0, \text{otherwise} \end{cases}, \quad h(t) = \begin{cases} t, 0 < t < 2T \\ 0, \text{otherwise} \end{cases}$$

$$t < 0, t > 3T \rightarrow x(\tau)h(t-\tau) = 0, \rightarrow y(t) = 0$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$

$$0 > t < T \rightarrow y(t) = \int_0^t (t-\tau)\mathrm{d}\tau = (t\tau - \tau^2)\Big|_0^t = t^2 - \frac{1}{2}t^2 = \frac{1}{2}t^2$$

$$T > t < 2T \rightarrow y(t) = \int_0^T (t-\tau)\mathrm{d}\tau = (t\tau - \tau^2)\Big|_0^T = \mathrm{T}t - \frac{1}{2}T^2$$

$$2T > t < 3T \rightarrow y(t) = \int_{t-2T}^{T} (t-\tau)\mathrm{d}\tau = (t\tau - \tau^2)\Big|_{t-2T}^{T} = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2$$

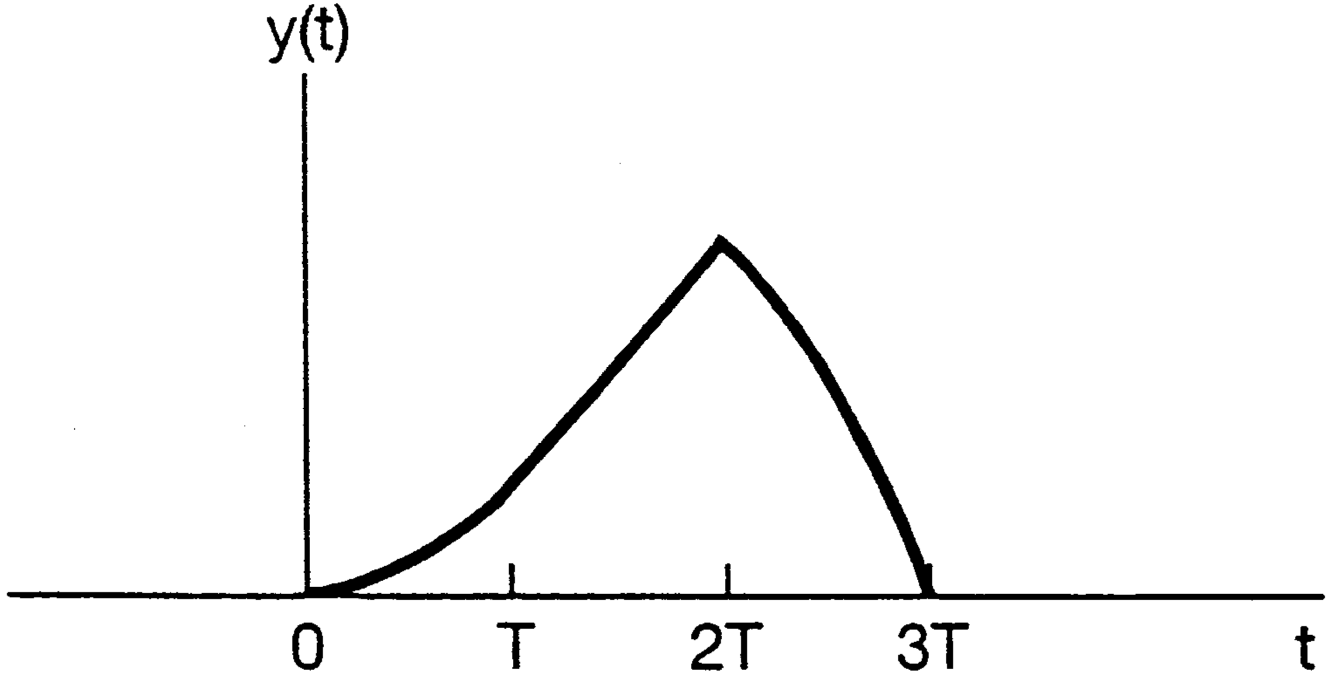


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Example 2.8

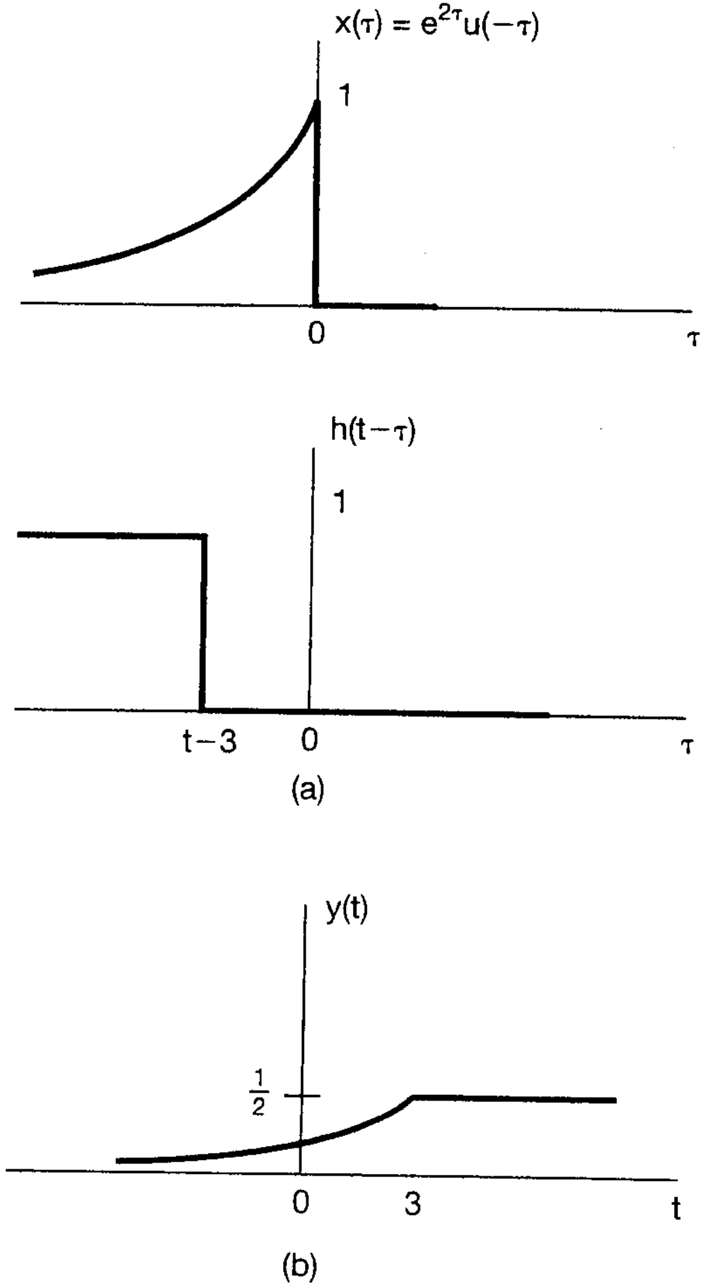


Figure 2.22 The convolution problem considered in Example 2.8.

$$x(t) = e^{2t}u(-t)$$

$$x(\tau) = e^{2\tau}u(-\tau)$$

$$h(t) = u(t-3)$$

$$h(\tau) = u(\tau-3),$$

$$h(-\tau) = u(-\tau-3)$$

$$h(t-\tau) = u(t-\tau-3)$$

$$= u(\underline{t-3}-\tau)$$

Example 2.8

$$x(t) = e^{2t}u(-t)\ ,\ \mathrm{h}(t) = u(t-3)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$t-3 \leq 0 \rightarrow y(t) = \int_{-\infty}^{t-3} e^{2\tau}d\tau = \frac{1}{2}e^{2(t-3)}$$

$$t-3 \geq 0 \rightarrow y(t) = \int_{-\infty}^{0} e^{2\tau}d\tau = \frac{1}{2}$$

2.3 Properties of LTI Systems

- Convolution sum and convolution integral:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- The properties (characteristics) hold in general only for LTI system. The unit impulse response of a ***nonlinear*** system ***does not*** completely characterize the behavior of the system.

2.3 Properties of linear time-invariant (LTI) systems

2.3.1 The commutative property:

In discrete time

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k], \quad (2.43)$$

In continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\mathrm{d}\tau. \quad (2.44)$$

2.3.2 The distributive property

In discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n], \quad (2.46)$$

And in continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \quad (2.47)$$

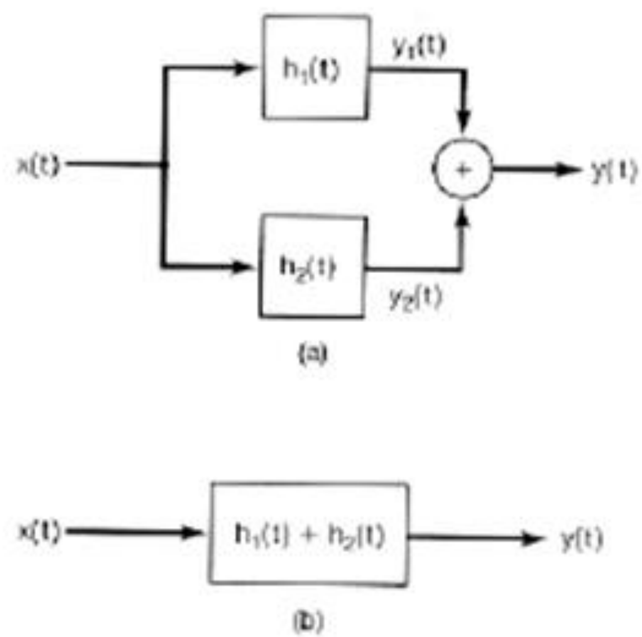


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

Example 2.10

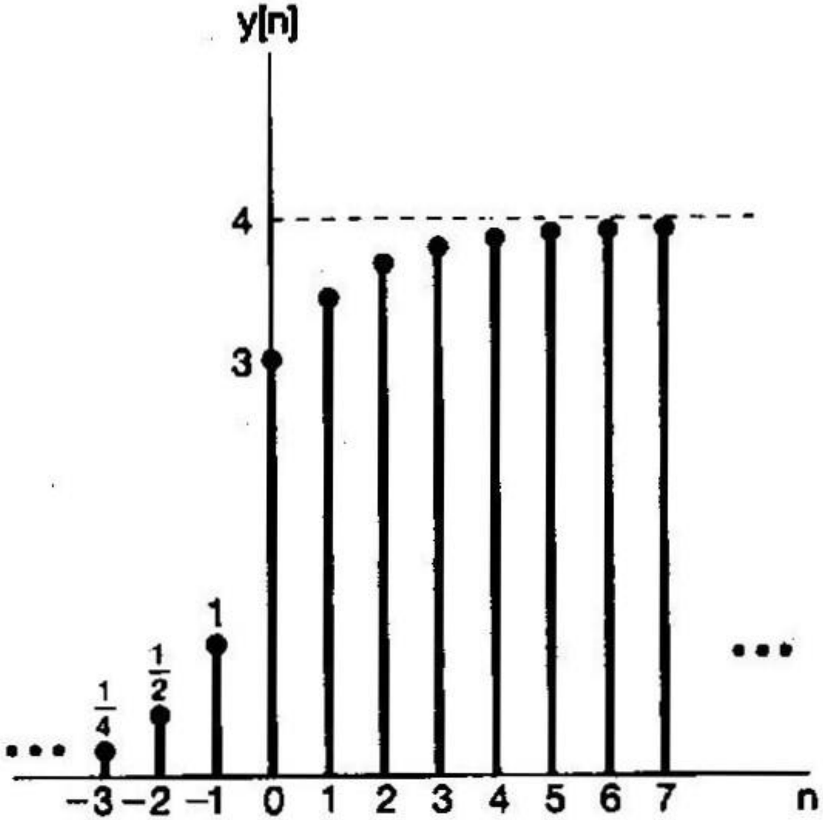


圖 2.24 例題 2.10 的信號 $y[n]=x[n]*h[n]$

$$y[n]=x[n]*h[n]$$

$$x[n]=(1/2)^n u[n]+2^n u[-n],$$

$$=x_1[n]+x_2[n]$$

$$h[n]=u[n]$$

$$y[n]=(x_1[n]+x_2[n])*h[n]$$

$$=x_1[n]*h[n]+x_2[n]*h[n]$$

$$=y_1[n]+y_2[n]$$

2.3.3 The associative property

In discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]. \quad (2.58)$$

In continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t). \quad (2.59)$$

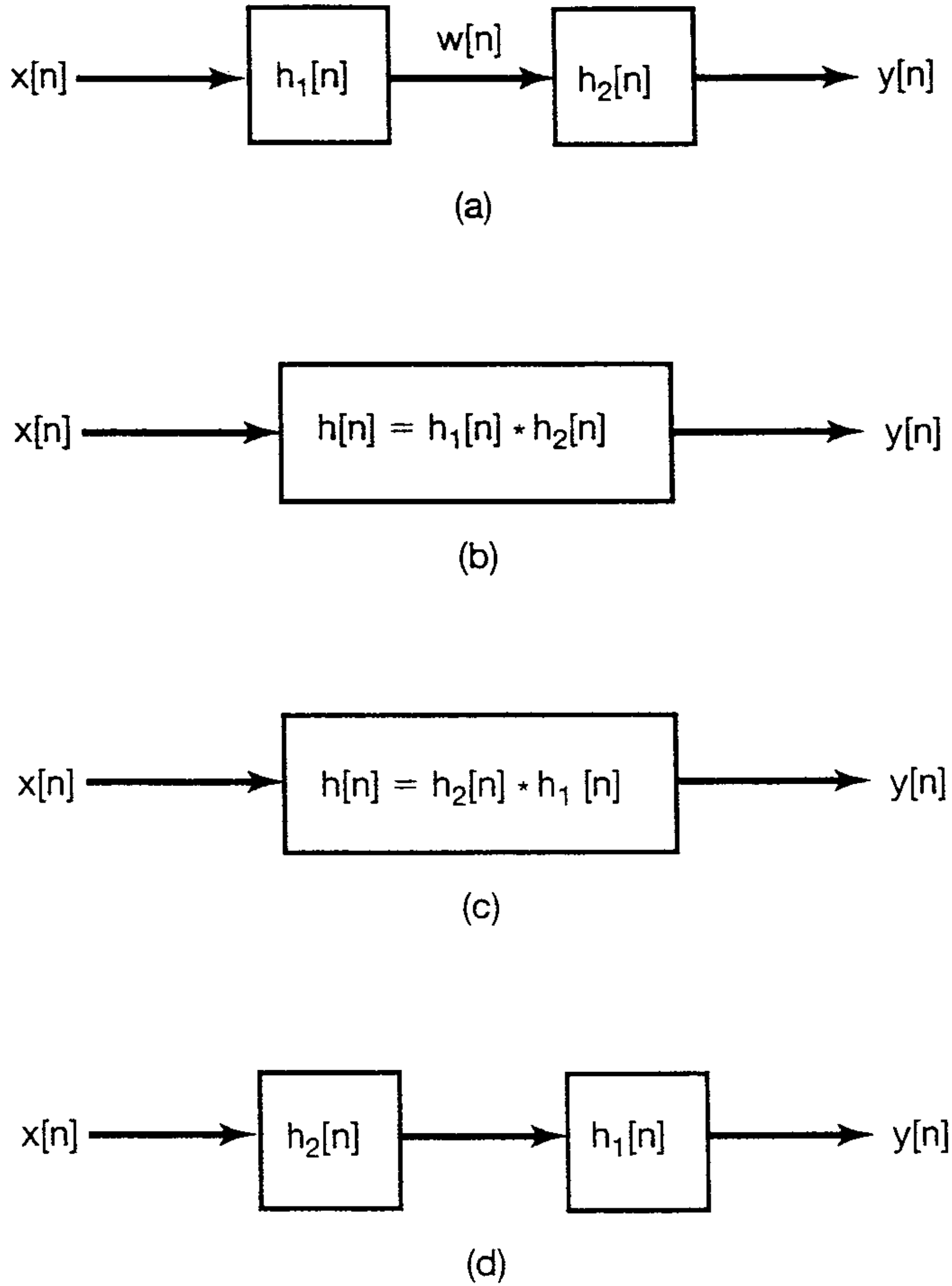


Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

2.3.4 LTI system with and without memory

In particular, a continuous-time LTI system is memoryless if $h(t)=0$ for $t\neq 0$, and such a memoryless LTI system has the form

$$y(t) = Kx(t) \qquad (2.46)$$

For some constant K and has the impulse response

$$h(t) = K\delta(t). \qquad (2.65)$$

2.3.5 Invertibility of LTI systems

The system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system. Furthermore, if an LTI system is invertible, then it has an LTI inverse.

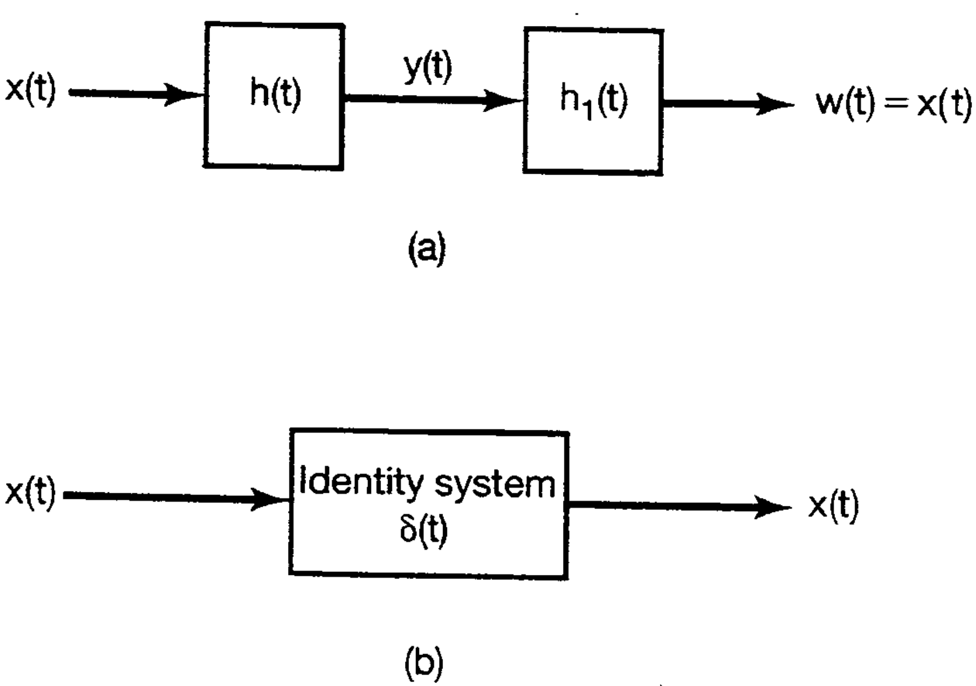


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

Example 2.11

- An LTI system: $y(t)=x(t-t_0)$ (2.68)
- Such a system is a delay if $t_0>0$ and is an advance if $t_0<0$.
- The impulse response for the system can be obtained from eq. (2.68) by taking the input to $\delta(t)$, i.e.

$$h(t)=\delta(t-t_0),\ x(t-t_0)=x(t)*\delta(t-t_0)$$

- If we take $h_1(t)=\delta(t+t_0)$, then

$$h(t)*h_1(t)=\delta(t-t_0)*\delta(t+t_0)=\delta(t)$$

Example 2.12

- Consider an LTI system with impulse response $h[n]=u[n]$.
- Using convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$

- Since $u[n\text{-}k]=0$ for $n\text{-}k<0$ and 1 for $n\text{-}k >=0$,

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

- This system is a summer or accumulator that computes the running sum of all the values of the input up to the present time.

- Such system is invertible, and its inverse is

  $w[n]=y[n]-y[n-1]=x[n]$,

  (analogy to $y[n]=x[n]-x[n-1]$)

  which is simply a first difference operation.

- Choosing $x[n]=\delta[n]$, we find that the impulse response of the inverse system is

$$h_1[n]=\delta[n]-\delta[n-1]$$

$$h[n]*h_1[n]=u[n]*\{\delta[n]-\delta[n-1]\}$$

$$=u[n]*\delta[n]-u[n]*\delta[n-1]$$

$$=u[n]-u[n-1]=\delta[n]$$

2.3.6 Causality for LTI systems

The output of a **causal system** depends only on the present and past values of the input to the system.

In order for a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k>n$. For this be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k>n$ must be zero. And

$$h[n] = 0 \quad \text{for } n < 0 \qquad (2.77)$$

For a causal **discrete-time** LTI system, the condition in eq. (2.77) implies that the convolution sum representation in eq. (2.6) and (2.7) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \qquad (2.78)$$

And the alternative equivalent form, eq.(2.43), becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k], \qquad (2.79)$$

Similarly, a **continuous-time** LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0, \qquad (2.80)$$

And in this case the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \quad (2.81)$$

- The pure time shift impulse with impulse response $h(t)=\delta(t-t_0)$ is ***causal*** for $\boldsymbol{t_0>=0}$, but is ***noncausal*** for $\boldsymbol{t_0<0}$
- Causality of an LTI system is equivalent to its ***impulse response being a causal signal***.

2.3.7 Stability for LTI systems

A system is stable if every bounded input produces a bounded output. Consider an input x[n] that is bounded in magnitude: $|x[n]|<B$, for all n. Then

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|,$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|,$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \qquad \text{For all } n. \qquad (2.85)$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad (2.86)$$

In continuous case,

$$|y(t)| = \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right| \leq B\int_{-\infty}^{+\infty}|h(\tau)|d\tau$$

$$\int_{-\infty}^{+\infty}|h(\tau)|d\tau < \infty. \qquad (2.87$$

Example 2.13

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \text{ and } \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

- Consider a system that is ***a pure time shift*** in either continuous time or discrete time.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| = 1$$

and

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{\infty} |\delta(\tau - t_0)|\, d\tau = 1$$

- Then we conclude that both of these systems are ***stable***.

Now consider the accumulator and integrator

- Impulse response functions $h[n] = u[n], \qquad h(t) = u(t)$

In discrete time, $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty$, not summable

and

in continuous time, $\int_{-\infty}^{\infty} h(\tau) d\tau = \int_{-\infty}^{\infty} |u(\tau)| d\tau = \int_{0}^{\infty} u(\tau) d\tau = \infty$,

$\therefore$ both systems are unstable!

2.3.8 The unit step response of an LTI system

$s[n] = u[n] * h[n] = h[n] * u[n]$

The unit step response, $s[n]$ or $s(t)$, corresponding to the output when $x[n]=u[n]$ or $x(t)=u(t)$. Then

$$s[n] = u[n] * h[n].$$

$s[n]$ is the unit step response of the accumulator.

Therefore,

$$s[n] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k]. \quad (2.91)$$

And $h[n]=s[n]-s[n-1]$. (2.92)

Similarly, in continuous time,

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \quad (2.93)$$

And $h(t) = \frac{ds(t)}{dt} = s'(t)$ (2.94)

- $s[n]$ can be seen as the response to the input $h[n]$ with unit step response $u[n]$
- Use the result shown in Ex. 2.12, we can get (2.91)

- In both continuous and discrete time, the ***unit step response*** can also be used to characterize an LTI system.

2.4 Causal LTI systems described by differential and difference equations

An extremely important class of both continuous-time systems and discrete-time systems is that for which the input and output are related through a *linear constant-coefficient differential equation*.

Because it’s too easy, we can study by ourselves without discuss on the class.

2.4.1 Linear constant-coefficient differential equations

- Example 2.14 $\frac{dy(t)}{dt} + 2y(t) = x(t)$

  $x(t)$: input

  $y(t)$: output

  $x(t)=Ke^{3t}u(t)$, where K is a real number

- A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a^k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

2.4.2 Linear constant-coefficient difference equations

- *N*th-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\}$$

- To calculate $y[n]$, we need to know $y[n-1]$, …, $y[n-N]$. It's a recursive equation.

Example 2.15

$$y[n]-\frac{1}{2}y[n-1]=x[n]\Rightarrow y[n]=x[n]+\frac{1}{2}y[n-1]$$

to begin the recursion, we need an initial condition.

consider $x[n]=K\delta[n], x[n]=0 \quad \text{for} \quad n\le -1$

$\Rightarrow y[n]=0 \;\text{ for } \quad n\le -1$

$$y[0]=x[0]+\frac{1}{2}y[-1]=K,\;\; y[1]=x[1]+\frac{1}{2}y[0]=\frac{1}{2}K$$

$$y[2]=x[2]+\frac{1}{2}y[1]=(\frac{1}{2})^2K$$

$$\Rightarrow y[n]=x[n]+\frac{1}{2}y[n-1]=(\frac{1}{2})^nK$$

Setting $K=1,\;\; x[n]=\delta[n].$ Thus $h[n]=(\frac{1}{2})^n u[n]$

$\Rightarrow$ An impules response of infinite duration

2.4.3 ***Block diagram representation*** of 1st order systems described by differential and difference equations

$$y[n]+ay[n-1]=bx[n], \Rightarrow y[n]=-ay[n-1]+bx[n]$$

Three basic operations for block diagram representation: ***addition***, ***multiplication*** by a coefficient, and ***delay*** (Fig. 2.27)

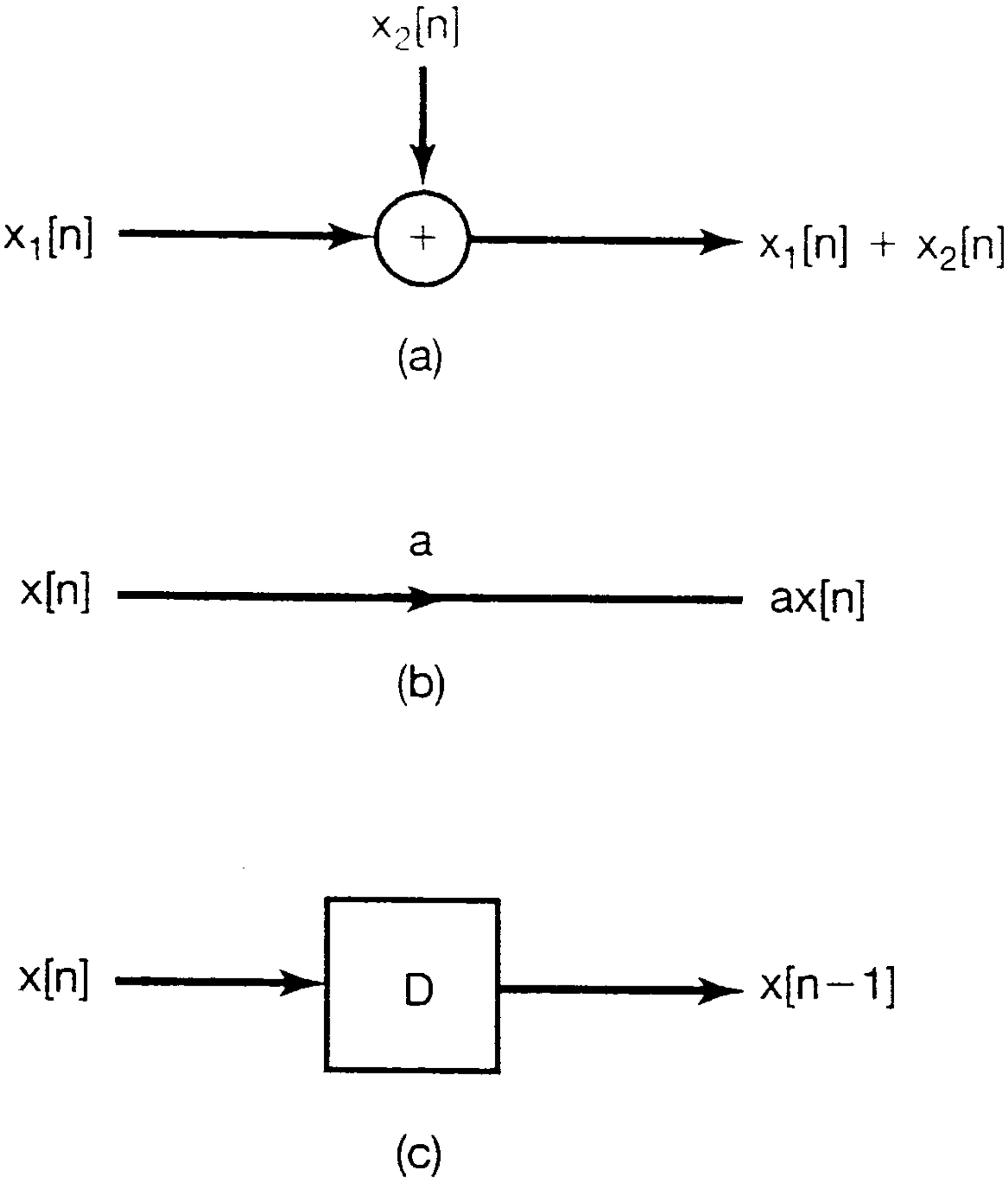


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

$$y[n]+ay[n-1]=bx[n], \Rightarrow y[n]=-ay[n-1]+bx[n]$$

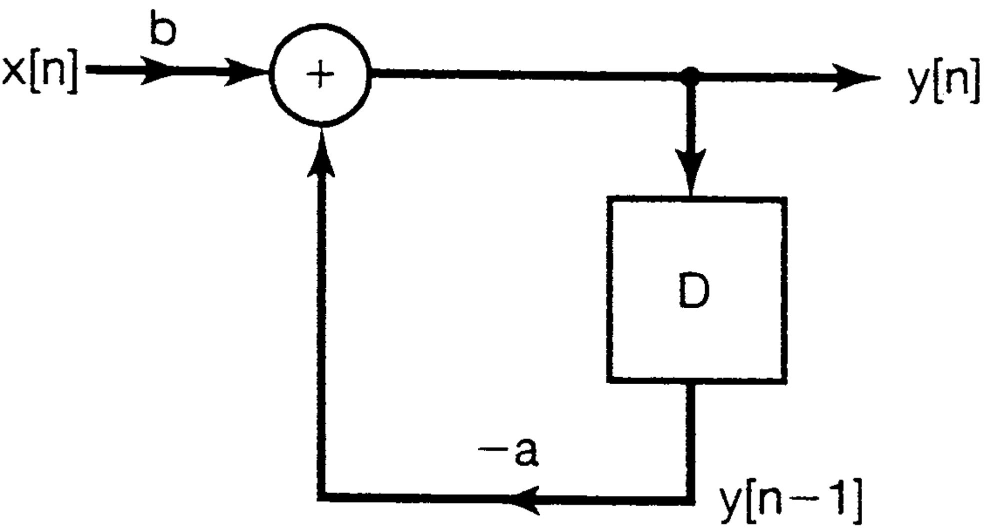


Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

Continuous-time system

$$\frac{dy(t)}{dt} + ay(t) = bx(t), \Rightarrow y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

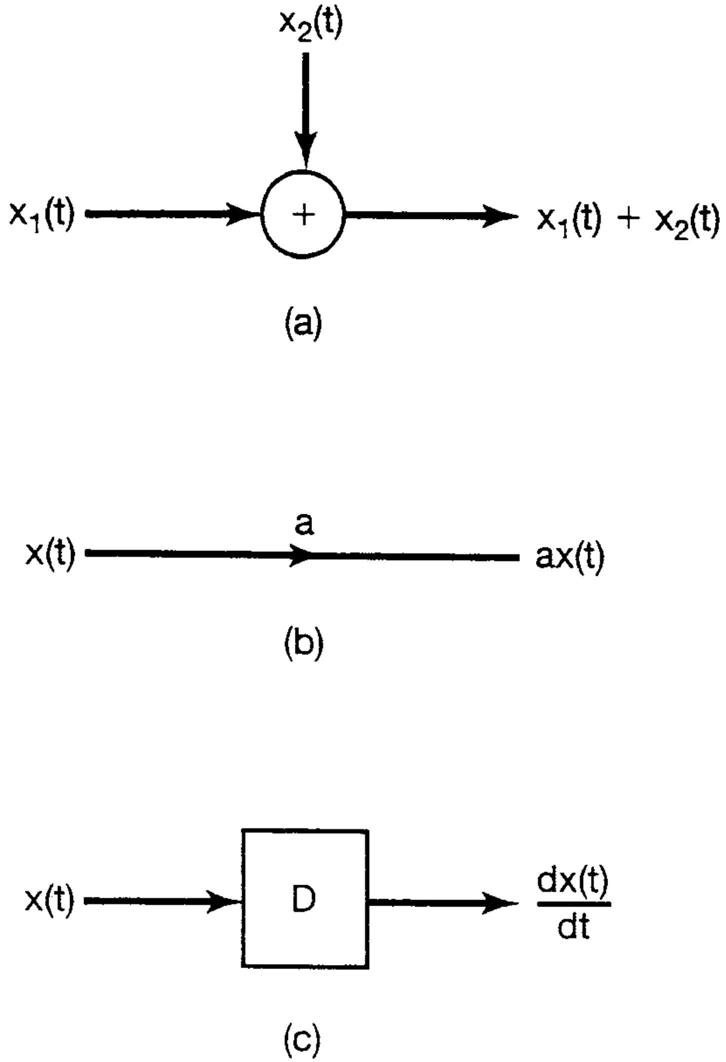


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

$$\frac{dy(t)}{dt}+ay(t)=bx(t),\Rightarrow y(t)=-\frac{1}{a}\frac{dy(t)}{dt}+\frac{b}{a}x(t)$$

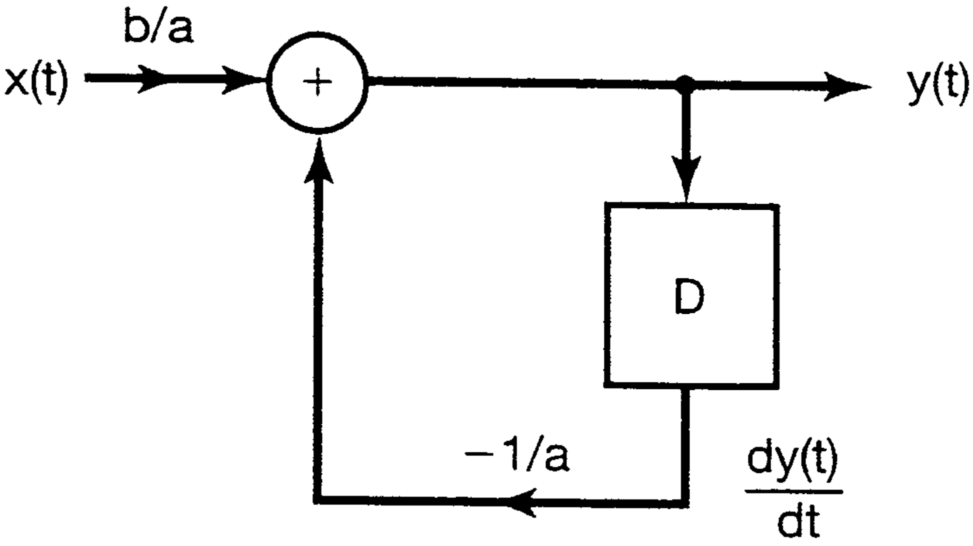


Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders multiplications by coefficients, and differentiators.

- Differentiators are both difficult to implement and extremely sensitive to errors and noise

$$\frac{dy(t)}{dt} = bx(t) - ay(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{t} \frac{dy(\tau)}{d\tau} = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

- In this form, our system can be implemented using the adder, coefficient multiplier, together with an integrator.

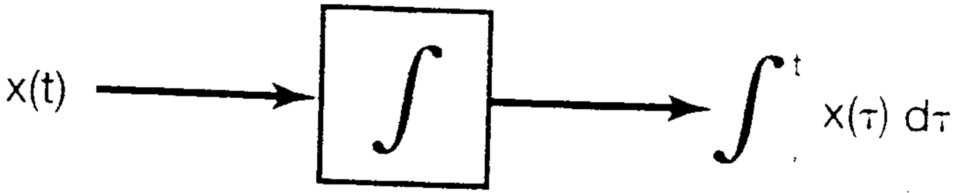


Figure 2.31 Pictorial representation of an integrator.

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau$$

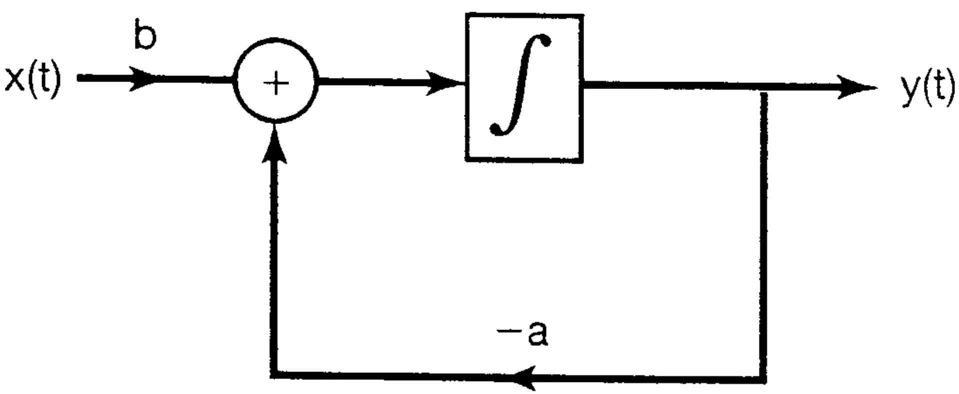


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

Season pass

- On the first day of college, the dean addressed the students, pointing out some of the rules:

  "The female dormitory will be out-of-bounds for all male students, and the male dormitory to the female students. Anybody caught breaking this rule will be fined $20 the first time. Anybody caught breaking this rule the second time will be fined $60. Being caught a third time will cost you $180. Are there any questions?"
  "How much for a season pass?"

2.5 Singularity functions (usually skip)

In particular, in section 1.4.2 we suggested that a continuous-time unit impulse could be viewed as the idealization of a pulse that is “short enough” so that its shape and duration is of no practical consequence---i.e., so that as far as the response of any particular LTI system is concerned, all of the area under the pulse can be thought of as having been applied instantaneously.

Following the example to show that the signals are in essence defined in terms of how they behave under convolution with other signals.

2.5.1 The unit impulse as an idealized short pulse

From the sifting property, eq. (2.27), the unit impulse $\delta(t)$ is the impulse response of the identity system.

That is,

$$x(t) = x(t) * \delta(t) \qquad (2.133)$$

For any signal x(t). Let $x(t)=\delta(t)$, we have

$$\delta(t) = \delta(t) * \delta(t) \qquad (2.134)$$

Suppose we think of $\delta(t)$ as the limiting from of a rectangular pulse. Let $\delta_{\Delta}(t)$ correspond to the rectangular pulse defined in figure 1.34, and let

$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t). \qquad (2.135)$$

where $r_{\Delta}(t)$ is as sketched in figure 2.33. If we wish to interpret $\delta(t)$ as the limit as $\Delta \to 0$ of $\delta_{\Delta}(t)$, then, by virtue of eq. (2.134), the limit as $\Delta \to 0$ for $r_{\Delta}(t)$ must also be a unit impulse.

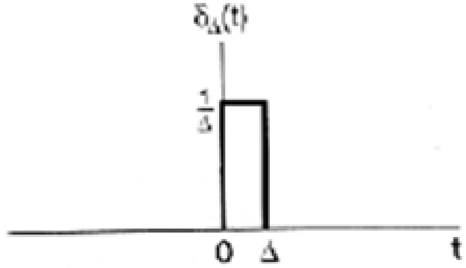


Figure 1.34 Derivative of $u_\Delta(t)$.

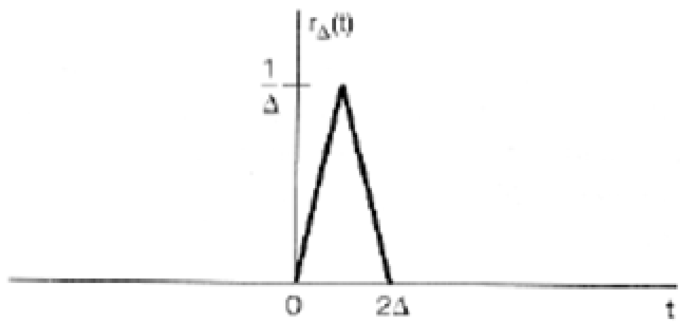


Figure 2.33 The signal $r_\Delta(t)$ defined in eq. (2.135).

The key words in the preceding paragraph are "behave like an impulse," where, as we have indicated, what we mean by this is that the response of an LTI system to all of these signals is essentially identical, as long as the pulse is "short enough," i.e., Δ is "small enough."

2.5.2 Defining the unit impulse through convolution

We define $\delta(t)$ as the signal for which

$$x(t) = x(t) * \delta(t) \qquad (2.138)$$

For any $x(t)$. If we replace $\delta(t)$ by any of these signals, then eq. (2.138) is satisfied in the limit. If we let $x(t)$=1 for all t, then

$$1 = \mathrm{x(t)} = \mathrm{x(t)} * \delta(t) = \delta(t) * \mathrm{x(t)}$$

$$= \int_{-\infty}^{+\infty} \boldsymbol{\delta(t)x(t-\tau)d\tau} = \int_{-\infty}^{+\infty} \boldsymbol{\delta(t)d\tau}.$$

so that the unit impulse has unit area.

2.5.3 Unit doublets and other singularity functions

The unit impulse is one of a class of signals known as singularity functions, each of which can be defined operationally in terms of its behavior under convolution. Consider

$$y(t) = \frac{dx(t)}{dt} \qquad (2.143)$$

The unit impulse response of this system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$. We have

$$\frac{dx(t)}{dt} = x(t) * u_1(t) \qquad (2.144)$$

Thereby,

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t). \qquad (2.145)$$

From eq. (2.144), we see that

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = x(t) * u_1(t) * u_1(t). \quad (2.146)$$

and therefore,

$$u_2(t) = u_1(t) * u_1(t). \qquad (2.147)$$

In general, $u_k(t)$, $k>0$, we have

$$u_k(t) = \underbrace{u_1 * \cdots * u_1(t)}_{k \text{ times}} \qquad (2.148)$$

In the other way round, by the complex equation's deduction we're not glad to see, we got the conclusion:

$$u_{-k}(t) = \underbrace{u * \cdots * u(t)}_{\text{k times}} = \int_{-\infty}^{t} u(\tau) d\tau. \qquad (2.157)$$

To use an alternative notation for $\delta(t)$ and u(t), namely,

$$\delta(t) = u_0(t) \qquad (2.159)$$

$$u(\mathrm{t}) = u_{-1}(t). \qquad (2.160)$$

Then, $u(t) * u_1(t) = \delta(t)$

$$u_{-1}(t) * u_1(t) = u_0(t) \qquad (2.161)$$

$$u_k(t) * u_{\mathrm{r}}(t) = u_{k+r}(t). \qquad (2.162)$$

2.6 summary

Through the chapter, we have learned about:

- Convolution integral
- Causality and Stability
- Linear constant-coefficient differential and difference equations
- Singularity functions