Signals and Systems

Chapter 3

Fourier Series for Periodic Signals

Instructor: Dr. Hsuan T. Chang

Department of Electrical Engineering
Yunlin University of Science and Technology

1

Introduction

- We explore an alternative representation (complex exponential) for signals and LTI systems.
- This chapter focuses on the representation of continuous-time and discrete-time signals referred to as the ***Fourier series***.
- If the input to an LTI system is expressed as a ***linear combination of periodic complex exponentials or sinusoids***, the output can also be expressed in this form, with ***coefficients*** that are related in a straightforward way to those of the input.

2

3.2 The Response of LTI Systems to Complex Exponentials

- The response of an LTI system to a complex exponential input is the same complex exponential with a change in amplitude:

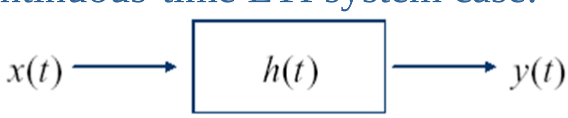
continuous time : $e^{st} \rightarrow H(s)e^{st}$

discrete time: $z^n \rightarrow H(z)z^n$

$H(s)$ is the system's eigenvalue.

e^{st} is the system's eigenfunction.

- Continuous-time LTI system case:



We consider $x(t)=e^{st}$,

then $y(t)=x(t)*h(t)$

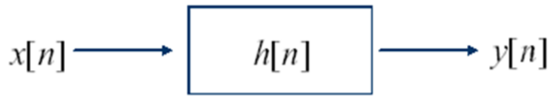
$$=\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$=\int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$=H(s)e^{st}$ where $H(s)=\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$

- Discrete-time LTI system case:



We consider $x[n]=z^n$,

then $y[n]=x[n]*h[n]$

$=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$=\sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$

$=z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$

$=H[z] \cdot z^n$

where $H[z] \equiv \sum_{k=-\infty}^{\infty} h[k]z^{-k}$

5

- Eigenfuction & Eigenvalue:

e^{st} is an eigenfunction, $H(s)$ is the corresponding eigenvalue

z^n is an eigenfunction, $H[z]$ is the corresponding eigenvalue

- Linear Combination:

If $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$

$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$

$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$

$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$

then $y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$

6

Specifically:

Table of the LTI system's linear combination

	Input	Output
CT	$x(t) = \sum_k a_k e^{s_k t}$	$y(t) = \sum_k a_k H(s_k) e^{s_k t}$
DT	$x[n] = \sum_k a_k z_k{}^n$	$y[n] = \sum_k a_k H(z_k) z_k{}^n$

7

Example 3.1

$y(t) = x(t-3), \qquad x(t) = e^{j2t}, \qquad s$
$= j2$

$$y(t) = e^{j2(t-3)} = e^{-j6} e^{j2t} = H(j2) e^{j2t}$$

Another way to find $H(s)$

$$h(t) = \delta(t-3)$$

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau - 3) e^{-s\tau} d\tau = e^{-3s} \Rightarrow H(s) = e^{-j6}$$

8

$$x(t) = \cos(4t) + \cos(7t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$$

$$y(t) = \cos[4(t-3)] + \cos[7(t-3)]$$

$$= \frac{1}{2}e^{-j12}e^{j4t} + \frac{1}{2}e^{j12}e^{-j4t} + \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t}$$

$$= \begin{cases} e^{j4t} \to s = j4, & H(j4) = \frac{1}{2}e^{-j12} \\ e^{-j4t} \to s = -j4, & H(-j4) = \frac{1}{2}e^{j12} \\ e^{j7t} \to s = j7, & H(j7) = \frac{1}{2}e^{-j21} \\ e^{-j7t} \to s = -j7, & H(-j7) = \frac{1}{2}e^{j21} \end{cases}$$

9

3.3 Fourier Series Representation of Continuous-time Periodic Signals

- Harmonically related complex exponentials:

$$\emptyset_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \qquad k = 0, \mp 1, \mp 2, \ldots\ldots$$

fundamental frequency: ω_0

fundamental period: $T = 2\pi/\omega_0$

- Linear Combination of harmonically related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

➔ is also periodic with period ***T***.

10

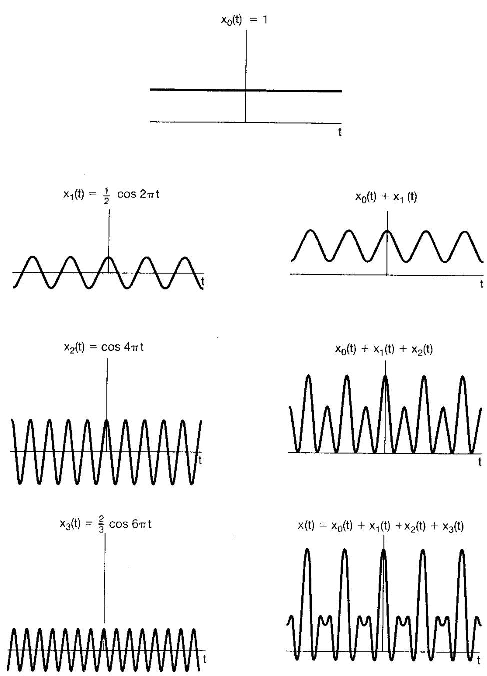
Example 3.2

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}\text{, where } a_0 = 1, a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2}, \qquad a_3 = a_{-3} = \frac{1}{3}$$

$$x(t)$$
$$= 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t}$$
$$+ e^{-j6\pi t})$$
$$= 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

11



12

Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

- Suppose that $x(t)$ is ***a real periodic*** signal, then $x^*(t)=x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{+\infty} a_k{}^* e^{-jk\omega_0 t} \text{ (replace } k \text{ by } -k)$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}{}^* e^{jk\omega_0 t}$$

$$\Rightarrow a_{-k}{}^* = a_k \Rightarrow a_k{}^* = a_{-k}$$

- $x(t)$ can be expressed in another form:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}] \qquad (a_{-k} = a_k{}^*)$$

$$= a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k{}^* e^{-jk\omega_0 t}]$$

$$= a_0 + \sum_{k=1}^{\infty} 2Re\{a_k e^{jk\omega_0 t}\}$$

13

- a_k is expressed in ***polar form***:

$$a_k = A_k e^{j\theta_k}$$

$$=> x(t) = a_0 + \sum_{k=1}^{\infty} 2Re\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

- a_k is expressed in ***rectangular form***:

$$a_k = B_k + jC_k$$

$$=> x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$$

- If a_k's are all real, $a_k = A_k = B_k, C_k = 0, \theta_k = 0$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} B_k \cos k\, w_0 t$$

14

3.3.2 Determination of the Fourier series representation of a continuous-time periodic signal

- Multiplying both sides of $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ by $e^{-jn\omega_0 t}$

$$=> x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating both sides from 0 to $T = {}^{2\pi}/_{\omega_0}$

$$=> \int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \cdot \int_0^T e^{j(k-n)\omega_0 t}dt$$

- If k=n, $\int_0^T e^{j(k-n)\omega_0 t}dt$= $\int_0^T 1dt = T$

If $k \neq n$,

- $\int_0^T e^{j(k-n)\omega_0 t}dt$, ω_0=$\frac{2\pi}{T}$

$= \frac{1}{j(k-n)\omega_0} e^{j(k-n)\omega_0 t}|_0^T$

$= \frac{1}{j(k-n)\omega_0} [e^{j(k-n)\omega_0 T}$- $e^0] = \ldots = 0$

- $e^{j(k-n)\omega_0 T}$- e^0

= cos$[(k-n)2\pi]$+$j$sin $[(k-n)2\pi]$-1

= 1+0-1

=0

$$=> \int_0^T x(t)e^{j(k-n)\omega_0 t}dt = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$$

we can find: $\int_0^T x(t)e^{-jn\omega_0 t}dt = a_n \cdot T$

$$=> a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

- For any interval of length T, we'll obtain the same result:

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

17

Definition 3.1 Fourier Series

for a continuous and periodic signal $x(t) = x(t+T)$

synthesis: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$

analysis: $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$

$\int_T$ is the integration from t_0 to t_0+T and t_0 can be any value

Specially, if $t_0 = 0$,

analysis: $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jk(2\pi/T)t}dt$

P.216

analysis: $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

$\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.

Specially, when $k = 0$,

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

P.216

Supplement: Alternative Forms of the Fourier Series

synthesis: $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t}$$

analysis: $$a_k = \frac{1}{T}\int_T x(t)e^{-j2\pi k f_0 t}dt$$

where $f_0 = 1/T$.

synthesis: $$x(t) = \sqrt{\frac{1}{T}}\sum_{k=-\infty}^{+\infty} a_k e^{-jk\omega_0 t}$$

analysis: $$a_k = \sqrt{\frac{1}{T}}\int_T x(t)e^{jk\omega_0 t}dt$$

Examples 3.3

$$x(t) = \sin w_0\, t = \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}, \qquad a_k = 0, \qquad \text{for } k$$
$$\neq +1 \text{ or } -1$$

It is easier to expand the sinusoidal signal as ***a linear combination of complex exponentials*** and identify the Fourier series coefficients ***by inspection***.

Examples 3.4

$$x(t) = 1 + \sin w_0\, t + 2\cos w_0\, t + \cos(2w_0 t + \frac{\pi}{4})$$
$$= 1 + \frac{1}{2j}[e^{jw_0 t} - e^{-jw_0 t}] + [e^{jw_0 t} + e^{-jw_0 t}] + \frac{1}{2}[e^{j(2w_0 t + \pi/4)} + e^{-j(2w_0 t + \pi/4)}]$$
$$= 1 + (1 + \frac{1}{2j})e^{jw_0 t} + (1 - \frac{1}{2j})e^{-jw_0 t} + (\frac{1}{2}e^{j(\pi/4)})e^{j2w_0 t} + (\frac{1}{2}e^{-j(\pi/4)})e^{-j2w_0 t}$$

$$a_0 = 1$$
$$a_1 = (1 + \frac{1}{2j}) = 1 - \frac{1}{2}j, \qquad a_{-1} = (1 - \frac{1}{2j})$$
$$= 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$
$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0 \text{ for } |k| > 2.$$

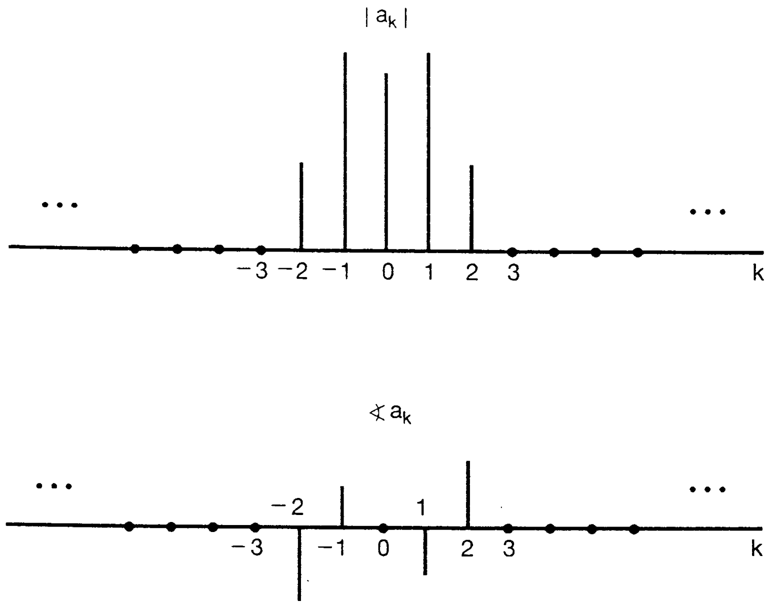


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Examples 3.5

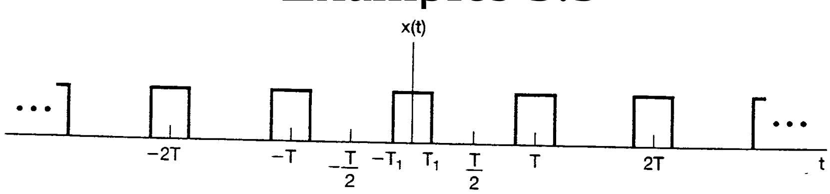


Figure 3.6 Periodic square wave.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

periodic with fundamental period T.

- Because $x(t)$ is symmetry about t=0, it is convenient to choose $-T/2 \le t < T/2$ as the interval over which the integration is performed.
- First, k=0 $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$
- For $k \neq 0$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0t}dt = -\frac{1}{jkw_0T}e^{-jkw_0t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{kw_0T}\left[\frac{e^{jkw_0T_1} - e^{-jkw_0T_1}}{2j}\right] = \frac{2\sin(kw_0T_1)}{kw_0T}$$

$$= \frac{\sin(kw_0T_1)}{k\pi}, \qquad k \neq 0$$

26

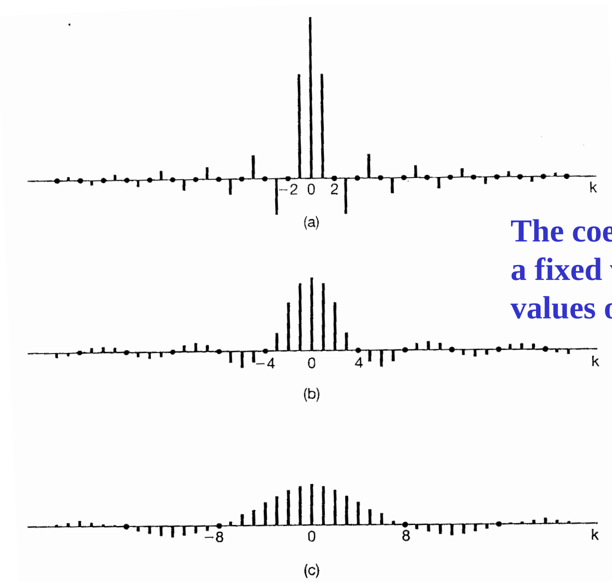


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

The coefficients are plotted for a fixed value of T_1 and several values of T.

27

3.4 Convergence of the Fourier Series

Examine the problem of approximating a given periodic signal $x(t)$ by a linear combination of a ***finite number*** of harmonically related complex exponentials.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}. \tag{3.47}$$

Let $e_N(t)$ denote the approximation error; that is,

$$e_N(t) = x(t) - xN(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.48}$$

In order to determine how good any particular approximation is, we need to specify a quantitative measure of the size of the approximation error. The criterion that we will use is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 dt. \tag{3.49}$$

As shown in Problem 3.66, the particular choice for the coefficients in eq. (3.47) that minimize the energy in the error is (the same as Eq. (3.39))

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt. \tag{3.50}$$

- The best approximation using only a finite number of harmonically related complex exponentials is obtained by truncating the Fourier series to the desired number of terms.
- As N increases, new terms are added and E_n decreases. If $x(t)$ has a Fourier series representation, then the limit of E_n as $N \rightarrow \infty$ is zero.

- Not any periodic signal has a Fourier series representation.

- What kind of signal can be represented by Fourier series:
  1. A signal has **finite energy** over a period

$$\int_T |x(t)|2\, dt < \infty \qquad (3.51)$$

When this condition is satisfied, we are guaranteed that a_k obtained from (3.39) are finite.

30

An alternative set of conditions guarantees that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. ➔ Dirichlet conditions

- Dirichlet conditions:

➢ Condition 1: $x(t)$ must be absolutely integrable over any period.

$$\int_T |x(t)|\, dt < \infty, \text{ then } a_k < \infty$$

➢ Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

➢ Condition 3: In any finite interval of time, there are a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

31

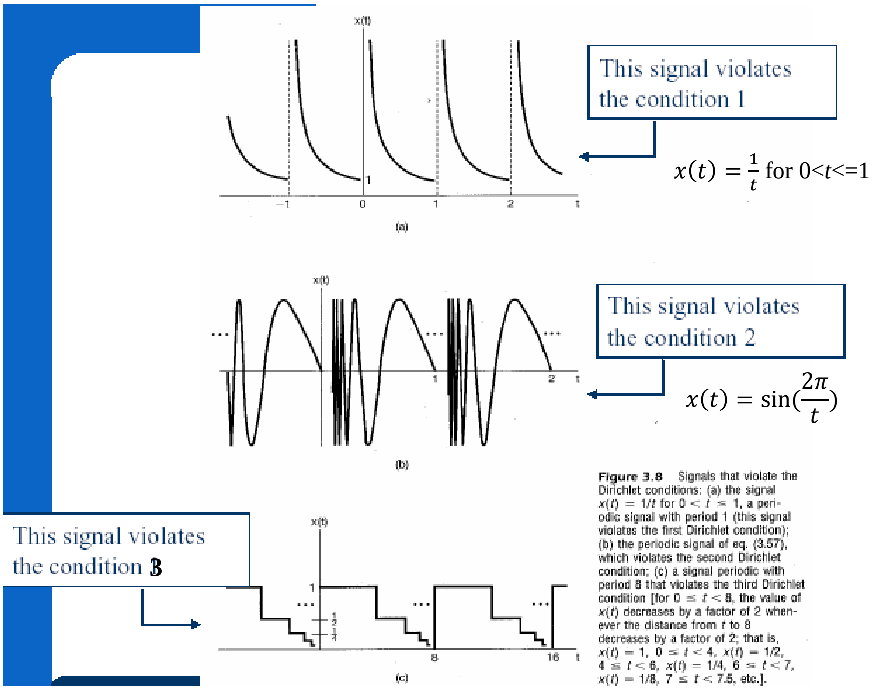
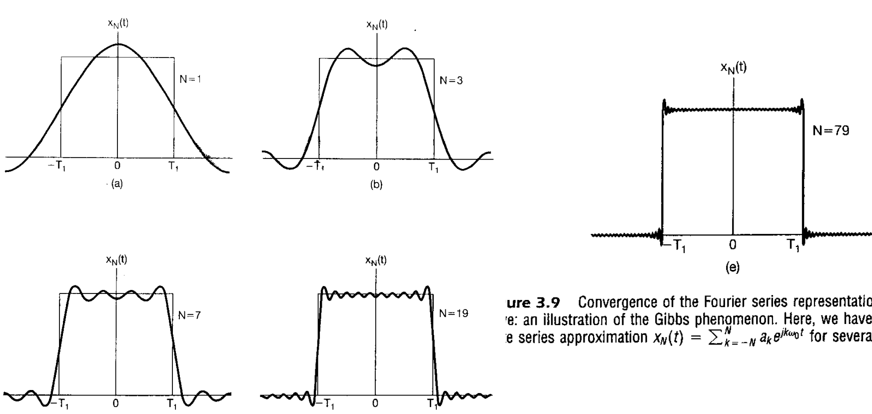


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

***Gibbs phenomenon*: the truncated Fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities.**



ure 3.9 Convergence of the Fourier series representation of a square 'e: an illustration of the Gibbs phenomenon. Here, we have depicted the e series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

33

3.5 Properties of Continuous-Time Fourier Series

3.5.1 Linearity

- 1. Linearity:

if

$$x(t) \overset{F.S.}{\longleftrightarrow} a_k$$

$$y(t) \overset{F.S.}{\longleftrightarrow} b_k$$

then

$$z(t) = Ax(t) + By(t) \overset{F.S.}{\longleftrightarrow} c_k = Aak + Bbk$$

34

3.5.2 Time Shifting & 3.5.3 Time Reversal

- 2. Time Shifting:

if $x(t) \overset{F.S.}{\longleftrightarrow} a_k$

then

$$x(t-t_0) \overset{F.S.}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

- 3. Time Reversal:

if $x(t) \overset{F.S.}{\longleftrightarrow} a_k$

then

$$x(-t) \overset{F.S.}{\longleftrightarrow} a_{-k}$$

35

3.5.4 Time Scaling & 3.5.5 Multiplication

- 4. Time Scaling:

if

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

- 5. Multiplication:

if $x(t) \overset{F.S.}{\longleftrightarrow} a_k$ $y(t) \overset{F.S.}{\longleftrightarrow} b_k$

then

$$x(t)y(t) \overset{F.S.}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

36

3.5.6 Conjugation and Conjugate Symmetry

- 6. Conjugation & Conjugate Symmetry:

*conjugation:

if $x(t) \overset{F.S.}{\longleftrightarrow} a_k$

then $x^*(t) \overset{F.S.}{\longleftrightarrow} a^*_{-k}$

*conjugate symmetry:

if $x(t)$ real, that is $x(t)=x^*(t)$

then $a_{-k} = a^*_k$

*if $x(t)$ real and even, then a_k real and even.

*If $x(t)$ real and odd, then a_k purely imaginary and odd.

37

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

- $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|2$

Average power in one period of $x(t)$

Note that:

$$\frac{1}{T}\int_T \left|\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right|^2 dt = \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_T |a_k|^2 \cdot 1 dt = \sum_{k=-\infty}^{\infty} |ak|2$$

*The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

38

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ } Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$a_k = a^*_{-k}$; $\mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\}$; $\mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\}$; $\|a_k\| = \|a_{-k}\|$; $\sphericalangle a_k = -\sphericalangle a_{-k}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]; $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{a_k\}$; $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

39

Example 3.6

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \quad (3.69)$$

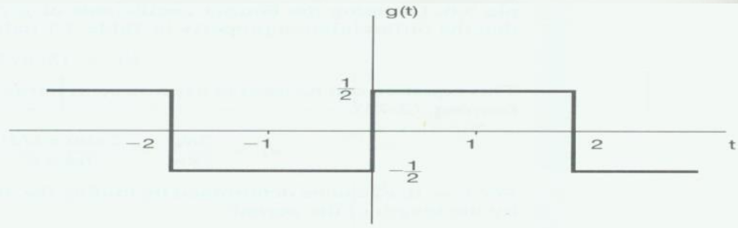


Figure 3.10 Periodic signal for Example 3.6.

40

$g(t)$ with $T = 4$ as shown in Fig. 3.10

$\rightarrow g(t) = x(t-1) - 1/2$

$\rightarrow b_k = a_k\, e^{-jk\pi/2}$

$$\rightarrow c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}$$

$$\rightarrow d_k = \begin{cases} a_k\, e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

$$\rightarrow d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

41

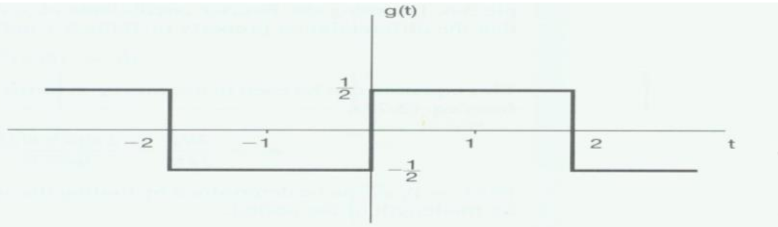


Figure 3.10 Periodic signal for Example 3.6.

The time-shift property in Table 3.1 indicates that, if the Fourier Series coefficients of $x(t)$ are denoted by a_k, the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}. \tag{3.70}$$

The Fourier coefficients of the *dc offset* in $g(t)$—i.e., the term $-1/2$ on the right-hand side of eq. (3.69)—are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}. \tag{3.71}$$

Applying the linearity property in Table 3.1, we conclude that the coefficients for $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from eqs. (3.45) and (3.46), yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}. \tag{3.72}$$

42

Example 3.7

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$ shown in Figure 3.11. The derivative of this signal is the signal $g(t)$ in Exam-

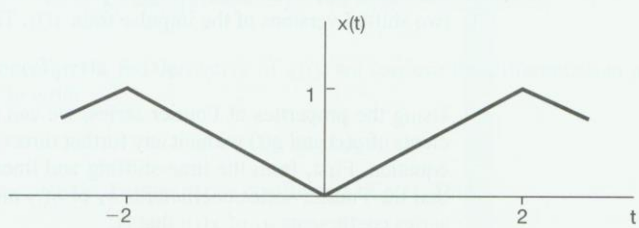


Figure 3.11 Triangular wave signal in Example 3.7.

43

Example 3.7

$x(t)$ with $T = 4$ and $\omega_0 = \frac{\pi}{2}$

as shown in Fig. 3.11

$$\rightarrow d_k = jk\left(\frac{\pi}{2}\right)e_k$$

$$\rightarrow e_k = \frac{2d_k}{jk\pi} = \frac{2\sin\left(\frac{\pi k}{2}\right)}{j(k\pi)^2}e^{-\frac{jk\pi}{2}}, \qquad k \neq 0$$

$$\rightarrow \text{For } k = 0, e_0 = \frac{1}{2}$$

44

Example 3.8

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or *impulse train.* This signal and its representation in terms of complex exponentials will play an important role when we discuss the topic of sampling in Chapter 7. The impulse train with period T may be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT); \tag{3.75}$$

it is illustrated in Figure 3.12(a). To determine the Fourier series coefficients a_k, we use eq. (3.39) and select the interval of integration to be $-T/2 \leq t \leq T/2$, avoiding the placement of impulses at the integration limits. Within this interval, $x(t)$ is the same as $\delta(t)$, and it follows that

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi t/T}dt = \frac{1}{T}. \tag{3.76}$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k). This is to be expected, since, according to Table 3.1, any real and even signal (such as our impulse train) should have real and even Fourier coefficients.

The impulse train also has a straightforward relationship to square-wave signals such as $g(t)$ in Figure 3.6, which we repeat in Figure 3.12(b). The derivative of $g(t)$ is the signal $q(t)$ illustrated in Figure 3.12(c). We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t + T_1) - x(t - T_1). \tag{3.77}$$

Using the properties of Fourier series, we can now compute the Fourier series coefficients of $q(t)$ and $g(t)$ without any further direct evaluation of the Fourier series analysis equation. First, from the time-shifting and linearity properties, we see from eq. (3.77) that the Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$b_k = e^{jk\omega_0 T_1}a_k - e^{-jk\omega_0 T_1}a_k.$$

Example 3.8

$x(t)$ with $T = 4$ and $\omega_0 = \frac{\pi}{2}$ as shown in Fig. 3.12

$$\rightarrow x(t) = \sum_{k=-\infty}^{\infty} \delta(t - KT);$$

$$\rightarrow a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-\frac{jk2\pi t}{T}} dt = \frac{1}{T};$$

$$\rightarrow q(t) = x(t + T_1) - x(t - T_1)$$

46

$$\rightarrow b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$\omega_0 = \frac{2\pi}{T} \rightarrow b_k = 2j\sin(k\omega_0 T_1) a_k = 2j\sin(k\omega_0 T_1)/T$$

$$b_k = jk\omega_0 c_k$$

$$\rightarrow c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

47

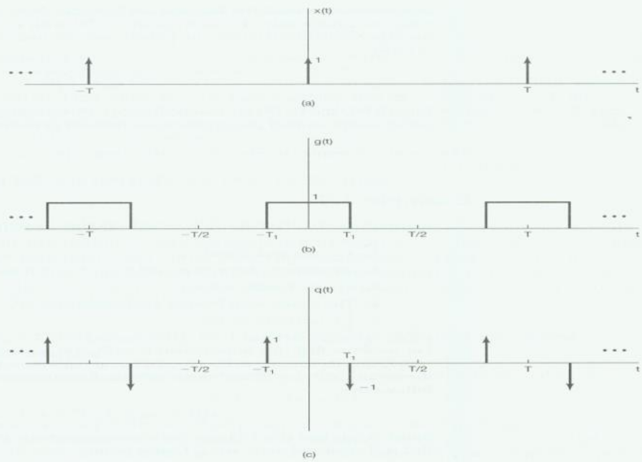


Figure 3.12 (a) Periodic train of impulses; (b) periodic square wave; (c) derivative of the periodic square wave in (b).

where $\omega_0 = 2\pi/T$. Using eq. (3.76), we then have

$$b_k = \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}.$$

Finally, since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property in Table 3.1 to write

$$b_k = jk\omega_0 c_k, \tag{3.78}$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.79}$$

Example 3.9

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2}a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

Let us show that this information is sufficient to determine the signal $x(t)$ to within a sign factor. According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_0, a_1, and a_{-1}. Then, since $x(t)$ has fundamental frequency $\omega_0 = 2\pi/4 = \pi/2$, it follows that

Fact 2

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1}e^{-j\pi t/2}.$$

Since $x(t)$ is real (Fact 1), we can use the symmetry properties in Table 3.1 to conclude that a_0 is real and $a_1 = a^*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}. \tag{3.81}$$

Let us now determine the signal corresponding to the Fourier coefficients b_k given in Fact 4. Using the time-reversal property from Table 3.1, we note that a_{-k} corresponds to the signal $x(-t)$. Also, the time-shift property in the table indicates that multiplication of the kth Fourier coefficient by $e^{-jk\pi/2} = e^{-jk\omega_0}$ corresponds to the underlying signal being shifted by 1 to the right (i.e., having t replaced by $t-1$). We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which, according to Fact 4 must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. From Table 3.1, it then follows that the Fourier coefficients of $x(-t+1)$ must be purely imaginary and odd. Thus, $b_0 = 0$ and $b_{-1} = -b_1$. Since time-reversal and time-shift operations cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is,

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2. \tag{3.82}$$

Example 3.9 –p2

We can now use Parseval's relation to conclude that

$$|b_1|^2 + |b_{-1}|^2 = 1/2. \qquad (3.83)$$

Substituting $b_1 = -b_{-1}$ in this equation, we obtain $|b_1| = 1/2$. Since b_1 is also known to be purely imaginary, it must be either $j/2$ or $-j/2$.

Now we can translate these conditions on b_0 and b_1 into equivalent statements on a_0 and a_1. First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-j\pi/2}b_{-1} = -jb_{-1} = jb_1$. Thus, if we take $b_1 = j/2$, then $a_1 = -1/2$, and therefore, from eq. (3.81), $x(t) = -\cos(\pi t/2)$. Alternatively, if we take $b_1 = -j/2$, then $a_1 = 1/2$, and therefore, $x(t) = \cos(\pi t/2)$.

50

Example 3.9

Fact 2: Fundamental Frequency: $\omega_0 = 2\pi/4 = \pi/2$

Fact 3: $x(t) = a_0 + a_1 e^{\frac{j\pi t}{2}} + a_{-1} e^{-\frac{j\pi t}{2}}$

Fact 1: $a_1 = a*_{-1}, a_1^* = a_{-1} \rightarrow x(t) = a_0 + 2\mathrm{Re}\left\{a_1 e^{\frac{j\pi t}{2}}\right\}$

Fact 4:

using odd signal, time reversal, and time shifting properties, b_k is corresponding to the signal

$$x(-(t-1)) = x(-t+1)$$

Since $x(t)$ is real, $x(-t+1)$ must also be real and its Fourier series also be purely imaginary and odd.

$b_0=0$, $b_1=-b_{-1}$ ➔ *purely imaginary*

51

Fact 5: $\frac{1}{4}\int_{4}|x(-t+1)|^2dt = \frac{1}{2}$

Use Parseval's relation

➔ $|b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$, $2|b_1|^2 = 1/2$,

➔ $b_1 = j/2$ or $-j/2$

➔ determine a_0, a_1, a_{-1} using Factor 4

➔ Determine $x(t)$ using Eq.(3.81)

52

- $b_1 = j/2$, $b_{-1} = -j/2$,
- $a_1 = e^{-\frac{j\pi}{2}} b_{-1} = -j\, b_{-1} = -1/2$
- $x(t) = 2\mathrm{Re}\left\{a_1 e^{\frac{j\pi t}{2}}\right\} = -\cos(\pi t/2)$

- $b_1 = -j/2$, $b_{-1} = j/2$,
- $a_1 = e^{\frac{j\pi}{2}} b_{-1} = j\, b_{-1} = 1/2$
- $x(t) = 2\mathrm{Re}\left\{a_1 e^{\frac{j\pi t}{2}}\right\} = \cos(\pi t/2)$

53

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

- The Fourier series representation of a discrete-time periodic signal is a ***finite series***, as opposed to the ***infinite series*** representation required for continuous-time signals.
- 連續時間信號的傅立葉級數表示是**無窮多**項數的，相反的，離散時間信號的傅立葉級數表示是**有限**項數的

54

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

- A discrete-time signal is ***periodic*** with period N

$$x[n] = x[n+N] \qquad (3.84)$$

- Fundamental frequency:

$$\omega_0 = 2\pi/N$$

- The set of all discrete-time complex exponential signals with period N

$$\emptyset_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, k = 0, \pm 1, \pm 2, \dots. \qquad (3.85)$$

$$\emptyset_0[n] = \emptyset_N[n]$$

$$\emptyset_1[n] = \emptyset_{N+1}[n] \Rightarrow \emptyset_k[n] = \emptyset_{k+rN}[n] \qquad (3.86)$$

55

- More general periodic sequences: **linear combination** of the sequence $\phi_k[n]$.

$$x[n] = \sum_{k} a_k \, \varphi_k[n] = \sum_{k} a_k \, e^{jk\omega_0 n} = \sum_{k} a_k \, e^{jk(2\pi/N)n}$$

- Since $\phi_k[n]$ are ***distinct only*** over a range of N successive values of k, the summation need only includes terms over this range: $k=<N>$

$$x[n] = \sum_{k=<N>} a_k \, \varphi_k[n] = \sum_{k=<N>} a_k \, e^{jk\omega_0 n} = \sum_{k=<N>} a_k \, e^{jk(2\pi/N)n}$$

56

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

- The discrete-time Fourier series representation of a periodic signal:

$$x[n] = \sum_{k=<N>} a_k \emptyset_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad (3.88)$$

$k=0,1,\ldots,N-1$

$k=3,4,\ldots,N+2$

- How to determine a_k? Multiply both sides by $e^{-jr(2\pi/N)n}$ and summing over N times

$$\sum_{n=<N>} x[n] \, e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k \, e^{j(k-r)\left(\frac{2\pi}{N}\right)n} \qquad (3.91)$$

$$= \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} \qquad (3.92)$$

57

- $\sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$

$= \sum_{n=0}^{N-1} e^{-j(k-r)\left(\frac{2\pi}{N}\right)n}$

$= \frac{1-[e^{-j(k-r)\left(\frac{2\pi}{N}\right)n}]^N}{1-e^{-j(k-r)\left(\frac{2\pi}{N}\right)}} = \frac{1-[e^{-j(k-r)2\pi n}]}{1-e^{-j(k-r)\left(\frac{2\pi}{N}\right)}}$

$= \frac{1-1}{1-e^{-j(k-r)\left(\frac{2\pi}{N}\right)}} = 0$

58

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

- Note that:

$$\sum_{n=<N>} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \quad (3.90)$$

then we choose $k=r$

$\Rightarrow \sum_{n=<N>} x[n]\, e^{-jr\left(\frac{2\pi}{N}\right)n} = Na_r$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jr\left(\frac{2\pi}{N}\right)n} \quad (3.95)$$

59

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

- If we take k: $0 \rightarrow N$-1

=>$x[n] = a_0 \emptyset_0[n] + a_1 \emptyset_1[n] + \cdots + a_{N-1} \emptyset_{N-1}[n]$

if we take k:$1 \rightarrow N$

=>$x[n] = a_1 \emptyset_1[n] + a_2 \emptyset_2[n] + \cdots + a_N \emptyset_N[n]$ (3.96)

but, as we know: $\emptyset_0[n] = \emptyset_N[N]$ (3.97)

therefore: $a_0 = a_N$

- We conclude that

$a_k = a_{k+N}$

that is , a_k is periodic with period N (3.98)

60

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

- Summary for the DTFS:

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \quad (3.94)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \quad (3.95)$$

61

Example 3.10

Consider the signal

$$x[n] = \sin \omega_0 n, \tag{3.99}$$

which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3.3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \tag{3.100}$$

Comparing eq. (3.100) with eq. (3.94), we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \tag{3.101}$$

N=5

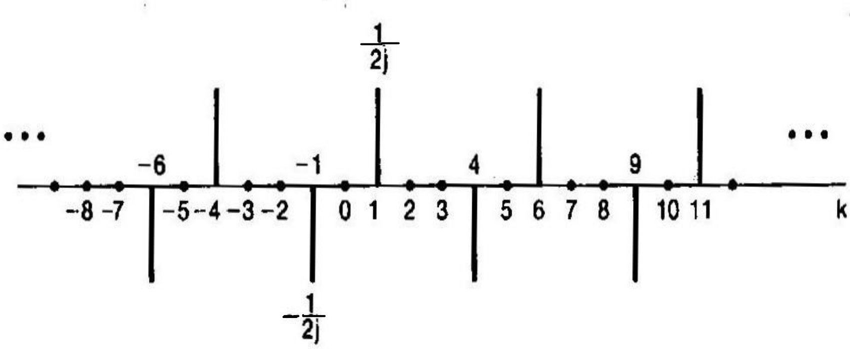


圖 3.13 $x[n] = \sin(2\pi/5)n$ 的傅立葉級數

Fourier Series Representation of Discrete-Time Periodic Signals 215

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).

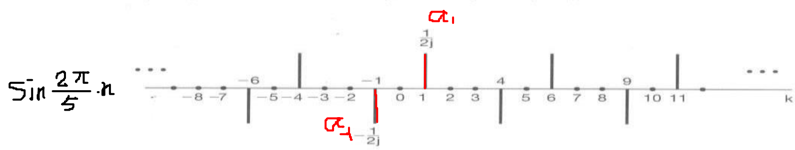


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n},$$

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure 3.14. Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

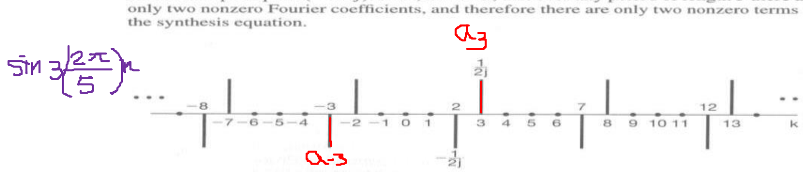


Figure 3.14 Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

64

Example 3.10

The signal $\rightarrow x[n] = \sin \omega_0 n$

$$\omega_0 = \frac{2\pi}{N}$$

$$x[n] = \frac{1}{2j}e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j}e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$a_1 = \frac{1}{2j} \;, \qquad a_{-1} = -\frac{1}{2j}$$

$$\omega_0 = \frac{2\pi M}{N} \rightarrow x[n] = \frac{1}{2j}e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j}e^{-jM\left(\frac{2\pi}{N}\right)n}$$

65

Example 3.11

$$x[n] = 1 + \sin(\frac{2\pi}{N})n + 3\cos(\frac{2\pi}{N})n + \cos(\frac{4\pi}{N}n + \frac{\pi}{2})$$

$$= 1 + \sin(\frac{2\pi}{N})n + 3\cos(\frac{2\pi}{N})n - \sin(\frac{4\pi}{N})n$$

$$= 1 + \frac{1}{2j}(e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}) + \frac{3}{2}(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}) - \frac{1}{2j}(e^{j\frac{4\pi}{N}n} - e^{-j\frac{4\pi}{N}n})$$

$$= 1 + (\frac{3}{2} + \frac{1}{2j})e^{j\frac{2\pi}{N}n} + (\frac{3}{2} - \frac{1}{2j})e^{-j\frac{2\pi}{N}n} - \frac{1}{2j}e^{j\frac{4\pi}{N}n} + \frac{1}{2j}e^{-j\frac{4\pi}{N}n}$$

$a_0 = 1 = a_N = a_{2N} = a_{3N} = \ldots..$

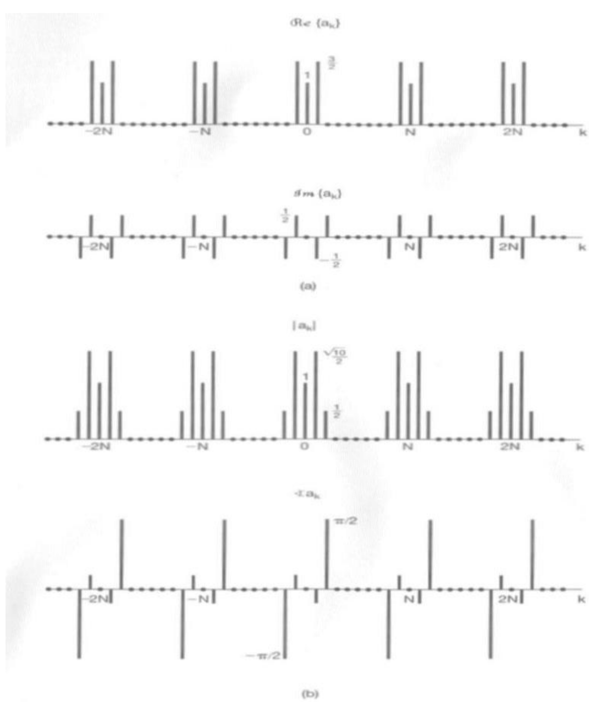
$a_1 = (\frac{3}{2} + \frac{1}{2j}) = a_{N+1} = a_{2N+1} = a_{3N+1} = \ldots,$ $a_{-1} = (\frac{3}{2} - \frac{1}{2j}) = a_{N-1}$

$= a_{2N-1} = a_{3N-1} = \ldots$

$a_2 = -\frac{1}{2j} = a_{N+2} = a_{2N+2} = a_{3N+2} = \ldots,$ $a_{-2} = \frac{1}{2j} = a_{N-2}$

$= a_{2N-2} = a_{3N-2} = \ldots$

66



67

Example 3.12 - The discrete-time periodic square wave

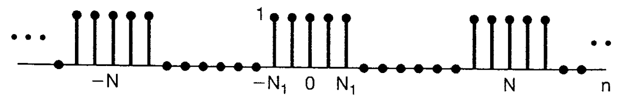


Figure 3.16 Discrete-time periodic square wave.

$x[n] = 1 \quad \text{for} \quad -N_1 \le n \le N_1$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

Let $m = n + N_1$,

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$$= \frac{1}{N}\frac{\sin[2\pi k(N_1+\frac{1}{2})/N]}{\sin(\pi k/N)}, \quad \text{for } k \ne 0, \pm N, \pm 2N, \ldots \quad (3.104)$$

See next page for details!

$$a_k = \frac{2N_1+1}{N}, \quad \text{for } k = 0, \pm N, \pm 2N, \ldots$$

$$a_k = \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$$= \frac{1}{N}e^{jk(2\pi/N)N_1} \times \frac{1-e^{-jk(2\pi/N)(2N_1+1)}}{1-e^{-jk(2\pi/N)}}$$

$$= \frac{1}{N}\frac{e^{jk(2\pi/N)N_1} - e^{-jk(2\pi/N)(N_1+1)}}{1-e^{-jk(2\pi/N)}}$$

$$= \frac{1}{N}\frac{e^{jk(2\pi/2N)}[e^{jk(2\pi/N)N_1} - e^{-jk(2\pi/N)(N_1+1)}]}{e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}}$$

$$= \frac{1}{N}\frac{e^{jk(2\pi/N)(N_1+\frac{1}{2})} - e^{-jk(2\pi/N)(N_1+\frac{1}{2})}}{e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}} = \frac{1}{N}\frac{2j\sin[2\pi k(N_1+\frac{1}{2})/N]}{2j\sin(\pi k/N)}$$

$$= \frac{1}{N}\frac{\sin[2\pi k(N_1+\frac{1}{2})/N]}{\sin(\pi k/N)}, \quad \text{for } k \ne 0, \pm N, \pm 2N, \ldots$$

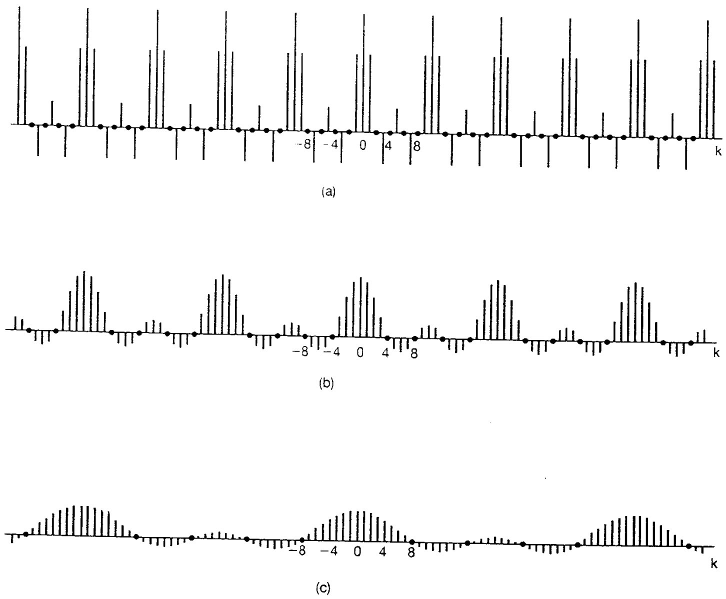


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

70

Comparison with continuous-time Fourier series

1. There are ***no convergence issues*** and there is ***no Gibbs phenomenon*** with the discrete-time Fourier series.
2. Any discrete-time periodic sequence $x[n]$ is completely specified by ***a finite number*** N of parameters.
3. The Fourier series analysis simply ***transform this set of N parameters into an equivalent set*** – the values of the N Fourier series coefficients.

71

3.7 Properties of Discrete-Time Fourier Series

- 1. Multiplication:

if

$$x[n] \overset{F.S.}{\longleftrightarrow} a_k$$

$$y[n] \overset{F.S.}{\longleftrightarrow} b_k$$

then

$$x[n]y[n] \overset{F.S.}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l b_{k-l} \quad (3.108)$$

A periodic convolution between the two periodic sequences of Fourier coefficients.

72

3.7.2 First Difference & 3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

- 2. First Difference:

if

$$x[n] \overset{F.S.}{\longleftrightarrow} a_k$$

then

$$x[n] - x[n-1] \overset{F.S.}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k \quad (3.109)$$

- 3. Parseval's Relation:

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{n=<N>} |a_k|^2 \quad (3.110)$$

- The left-side is the average power in one period of $x[n]$
- The right-side is the average power in all harmonic components of $x[n]$

73

Example 3.13

Let us consider the problem of finding the Fourier series coefficients a_k of the sequence $x[n]$ shown in Figure 3.19(a). This sequence has a fundamental period of 5. We observe that $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ in Figure 3.19(b) and the *dc sequence* $x_2[n]$ in Figure 3.19(c). Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k, we use the linearity property of Table 3.2 to conclude that

$$a_k = b_k + c_k. \quad (3.111)$$

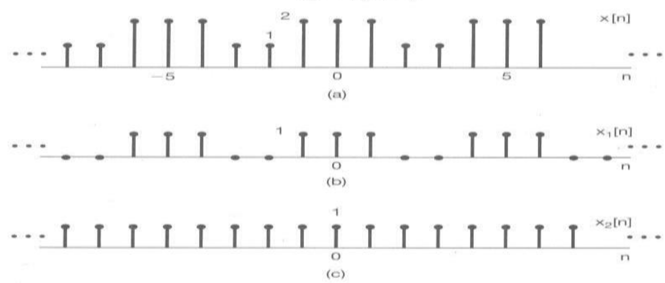


Figure 3.19 (a) Periodic sequence $x[n]$ for Example 3.13 and its representation as a sum of (b) the square wave $x_1[n]$ and (c) the dc sequence $x_2[n]$.

From Example 3.12 (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \quad (3.112)$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1. \quad (3.113)$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5. The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component. We can now substitute the expressions for b_k and c_k into eq. (3.111) to obtain

$$a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \quad (3.114)$$

74

Example 3.13

From Example 3.12(with $N_1 = 1$ & $N = 5$)

$x_1[n]$

$$\rightarrow b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, \quad \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$x_2[n] \rightarrow$ only have dc value

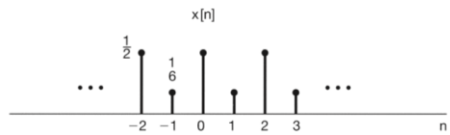
$$\rightarrow c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1$$

$$\rightarrow a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, \text{ for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{8}{5} \quad , \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

75

Example 3.14

The signal → $x[n]$:
1. periodic $N = 6$
2. $\sum_{n=0}^{5} x[n] = 2$
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions



Fact 2: $a_0 = \frac{1}{N}\sum_{k=<N>} x[n] = 2/6 = 1/3$

Fact 3: $a_3 = \frac{1}{N}\sum_{k=<N>} x[n]\, e^{-j3\omega_0 n} =$

$\frac{1}{N}\sum_{k=<N>} x[n]\, e^{-j3\left(\frac{2\pi}{6}\right)n} = \sum_{n=2}^{7} (-1)^n x[n]/6 = 1/6$

Fact 4: $P = \sum_{k=0}^{5} |a_k|^2$, to minimize the power, let $a_1 = a_2 = a_4 = a_5 = 0$

$$x[n] = a_0 + a_3 e^{j\pi n} = \left(\frac{1}{3}\right) + \left(\frac{1}{6}\right)(-1)^n$$

76

Example 3.14

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Let us determine the sequence $x[n]$. We denote the Fourier series coefficients of $x[n]$ by a_k. From Fact 2, we conclude that $a_0 = 1/3$. Noting that $(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$, we see from Fact 3 that $a_3 = 1/6$. From Parseval's relation (see Table 3.2), the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} |a_k|^2. \quad (3.115)$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and a_3 are prespecified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n, \quad (3.116)$$

which is sketched in Figure 3.20.

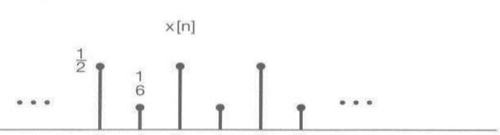


Figure 3.20 Sequence $x[n]$ that is consistent with the properties specified in Example 3.14.

77

Example 3.15

$x[n], y[n] \rightarrow$ period N

The siginal $\rightarrow w[n] = \sum_{r=<N>} x[r]y[n-r] \rightarrow$ perod of $N = 7$

$$\rightarrow c_k = \frac{\sin^2(\frac{3\pi k}{7})}{7\sin^2\left(\frac{\pi k}{7}\right)}$$

We observe that $c_k = 7d_k^2 = 7\, d_k \times d_k$

$$\rightarrow w[n] = \sum_{r=<7>} x[r]x[n-r] = \sum_{r=-3}^{3} x[r]x[n-r]$$

$$\rightarrow w[n] = \sum_{r=-3}^{3} \hat{x}[r]x[n-r] = \sum_{r=-\infty}^{+\infty} \hat{x}[r]x[n-r]$$

Example 3.15

Example 3.15

In this example we determine and sketch a periodic sequence, given an algebraic expression for its Fourier series coefficients. In the process, we will also exploit the periodic convolution property (see Table 3.2) of the discrete-time Fourier series. Specifically, as stated in the table and as shown in Problem 3.58, if $x[n]$ and $y[n]$ are periodic with period N, then the signal

$$w[n] = \sum_{r=\langle N\rangle} x[r]y[n-r]$$

is also periodic with period N. Here, the summation may be taken over any set of N consecutive values of r. Furthermore, the Fourier series coefficients of $w[n]$ are equal to Na_kb_k, where a_k and b_k are the Fourier coefficients of $x[n]$ and $y[n]$, respectively.

Suppose now that we are told that a signal $w[n]$ is periodic with a fundamental period of $N = 7$ and with Fourier series coefficients

$$c_k = \frac{\sin^2(3\pi k/7)}{7\sin^2(\pi k/7)}. \tag{3.117}$$

We observe that $c_k = 7d_k^2$, where d_k denotes the sequence of Fourier series coefficients of a square wave $x[n]$, as in Example 3.12, with $N_1 = 1$ and $N = 7$. Using the periodic convolution property, we see that

$$w[n] = \sum_{r=\langle 7\rangle} x[r]x[n-r] = \sum_{r=-3}^{3} x[r]x[n-r], \tag{3.118}$$

where, in the last equality, we have chosen to sum over the interval $-3 \le r \le 3$. Except for the fact that the sum is limited to a finite interval, the product-and-sum method for evaluating convolution is applicable here. In fact, we can convert eq. (3.118) to an ordinary convolution by defining a signal $\hat{x}[n]$ that equals $x[n]$ for $-3 \le n \le 3$ and is zero otherwise. Then, from eq. (3.118),

$$w[n] = \sum_{r=-3}^{3} \hat{x}[r]x[n-r] = \sum_{r=-\infty}^{+\infty} \hat{x}[r]x[n-r].$$

That is, $w[n]$ is the aperiodic convolution of the sequences $\hat{x}[n]$ and $x[n]$.

The sequences $x[r]$, $\hat{x}[r]$, and $x[n-r]$ are sketched in Figure 3.21 (a)–(c). From the figure we can immediately calculate $w[n]$. In particular we see that $w[0] = 3$; $w[-1] = w[1] = 2$; $w[-2] = w[2] = 1$; and $w[-3] = w[3] = 0$. Since $w[n]$ is periodic with period 7, we can then sketch $w[n]$ as shown in Figure 3.21(d). 30

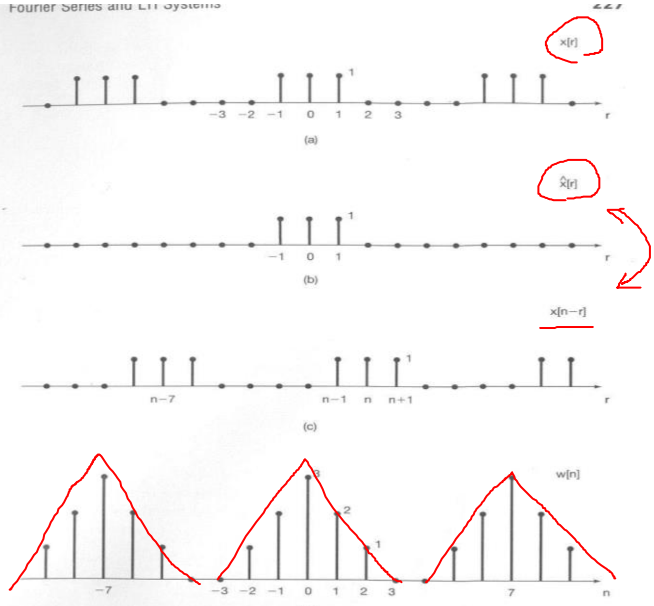


Figure 3.21 (a) The square-wave sequence $x[r]$ in Example 3.15; (b) the sequence $\hat{x}[r]$ equal to $x[r]$ for $-3 \le r \le 3$ and zero otherwise; (c) the sequence $x[n-r]$; (d) the sequence $w[n]$ equal to the periodic convolution of $x[n]$ with itself and to the aperiodic convolution of $\hat{x}[n]$ with $x[n]$.

81

When s or z are general complex numbers, $H(s)$ and $H(z)$ are referred to as the

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x[n]\\y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l}a_k\\b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic}\\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only}\\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

82

3.8 Fourier Series and LTI Systems

- In continuous time LTI system:

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

- From the beginning of Section 3.2, Eq. (3.6)

  if $x(t) = e^{st}$

  then $y(t) = H(s)e^{st}$

  where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

83

- In discrete-time LTI system:

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

From the beginning of Section 3.2, Eq. (3.10)

if $x[n] = z^n$

then $y[n] = H(z)\, z^n$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

Consider the C-T case: $s=jw$, $e^{st}=e^{jwt}$, $H(s)=H(jw)$

For D-T case: $|z|=1$, $z=e^{jw}$, $z^n=e^{jwn}$, $H(z)=H(e^{jw})$

- For $s = jw$ and $e^{-st} = e^{-jwt}$

$$H(s) = H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt \qquad (3.121)$$

For $z = e^{jw}$ and $z^n = e^{jwn}$

$$H(z) = H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn} \qquad (3.122)$$

3.8 Fourier Series and LTI Systems

- If $x(t)$ is a periodic signal then the FS representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

Using Eqs. (3-13) and (3-14) (p. 7 in PPT) and $s_k = jkw_0$

$$y(t) = \sum_{k=-\infty}^{\infty} \underline{a_k H(jkw_0)}\, e^{jkw_0 t} \qquad (3.123)$$

The Fourier series coefficients for y(t)

$y(t)$ is also periodic with the same fundamental frequency as $x(t)$

86

3.8 Fourier Series and LTI Systems

- If $x[n]$ is a periodic signal then the FS representation:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Using Eqs. (3-15) and (3-16) and $z_k = e^{jkw_0}$

$$y[n] = \sum_{k=<N>} \underline{a_k H\left(e^{jk\left(\frac{2\pi}{N}\right)}\right)}\, e^{jk\left(\frac{2\pi}{N}\right)n} \qquad (3.131)$$

The Fourier series coefficients for $y[n]$

$y[n]$ is also periodic with the same fundamental frequency as $x[n]$

87

Example 3.16

Example 3.16

Suppose that the periodic signal $x(t)$ discussed in Example 3.2 is the input signal to an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}, \text{ where } a_0 = 1, a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2}, \qquad a_3 = a_{-3} = \frac{1}{3}$$

88

$$h(t) = e^{-t}u(t)$$

$$\rightarrow H(j\omega) = \int_0^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau = -\frac{1}{1+j\omega}e^{-\tau}e^{-j\omega\tau}\Big|_0^{\infty} = \frac{1}{1+j\omega}$$

$$\omega_0 = 2\pi \rightarrow y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

$$b_k = a_k H(jk2\pi) \rightarrow b_0 = 1$$

$$\rightarrow b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right),\ b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right),$$

$$\rightarrow b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right),\ b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right)$$

$$\rightarrow b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right),\ b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right),$$

89

$$y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi kt + \theta_k) = 1 + 2\sum_{k=1}^{3}[E_k \cos 2\pi kt - F_k \sin 2\pi kt]$$

$$\left(b_k = D_k e^{j\theta_k} = E_k + jF_k, \qquad k = 1, 2, 3\right)$$

For example:

$$D_1 = |b_1| = \frac{1}{4\sqrt{1+4\pi^2}} \qquad \theta = \angle b_1 = -\tan^{-1}(2\pi)$$

$$E_1 = \Re e\{b_1\} = \frac{1}{4(1+4\pi^2)} \qquad F_1 = \Im m\{b_1\} = -\frac{\pi}{2(1+4\pi^2)}$$

$$\rightarrow x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\rightarrow y[n] = \sum_{k=<N>} a_k H\left(e^{\frac{j2\pi k}{N}}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

90

Example 3.17

Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$, and with the input

$$x[n] = \cos\left(\frac{2\pi n}{N}\right). \tag{3.132}$$

As in Example 3.10, $x[n]$ can be written in Fourier series form as

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}.$$

Also, from eq. (3.122),

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}\left(\alpha e^{-j\omega}\right)^n. \tag{3.133}$$

This geometric series can be evaluated using the result of Problem 1.54, yielding

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}. \tag{3.134}$$

Using eq. (3.131), we then obtain the Fourier series for the output:

$$\begin{aligned} y[n] &= \frac{1}{2}H\left(e^{j2\pi/N}\right)e^{j(2\pi/N)n} + \frac{1}{2}H\left(e^{-j2\pi/N}\right)e^{-j(2\pi/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}. \end{aligned} \tag{3.135}$$

91

Example 3.17 –p2

If we write

$$\frac{1}{1-\alpha e^{-j2\pi/N}} = re^{j\theta},$$

then eq. (3.135) reduces to

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right). \tag{3.136}$$

For example, if $N = 4$,

$$\frac{1}{1-\alpha e^{-j2\pi/4}} = \frac{1}{1+\alpha j} = \frac{1}{\sqrt{1+\alpha^2}} e^{j(-\tan^{-1}(\alpha))},$$

and thus,

$$y[n] = \frac{1}{\sqrt{1+\alpha^2}}\cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right).$$

92

Example 3.17

An LTI system (with $h[n] = \alpha^n u[n], -1 < \alpha < 1$)

input → $x[n] = \cos\left(\frac{2\pi n}{N}\right) = \frac{1}{2}e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}e^{-j\left(\frac{2\pi}{N}\right)n}$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

Output: $y[n] = \frac{1}{2}H\left(e^{\frac{j2\pi}{N}}\right)e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}H\left(e^{-\frac{j2\pi}{N}}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$

$$= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$

Let $\frac{1}{1-\alpha e^{-j2\pi/N}} = re^{j\theta} \rightarrow y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)$

If N=4, $\frac{1}{1-\alpha e^{-j2\pi/4}} = \frac{1}{1+\alpha j} = \frac{1}{\sqrt{1+\alpha^2}}e^{-j\tan^{-1}(\alpha)}$

$$\Rightarrow y[n] = \frac{1}{\sqrt{1+\alpha^2}}\cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right)$$

3.9 Filtering

- Filtering:
  1. ***change*** the relative amplitudes of the frequency components in a signal
  2. ***eliminate*** some frequency components entirely
- Frequency-***shaping*** filters: LTI systems that change the shape of the signal spectrum
- Frequency-***selective*** filters: systems that are designed to ***pass*** some frequencies essentially ***undistorted*** and ***significantly attenuate*** or ***eliminate*** others

94

3.9.1 Frequency shaping filters

- An application: audio systems
  - The frequency-shaping filters correspond to LTI systems whose frequency response can be changed by manipulating the tone controls.
  - Tone: **bass - low** frequency energy

    treble – high frequency energy
  - Equalizing filters are often included in the preamplifier to compensate for the frequency-response characteristics of the speakers

95

- Another class: for which the filter output is the derivative of the filter input.
- $y(t) = \frac{dx(t)}{dt}, x(t) = e^{j\omega t}, y(t) = j\omega e^{j\omega t}$
- $H(j\omega) = j\omega$
- A complex exponential input $e^{j\omega t}$ will receive greater amplification for large values of ω.
- Consequently, differentiating filters are useful in enhancing rapid variations or transitions in a signal.

96

Fig. 3.23

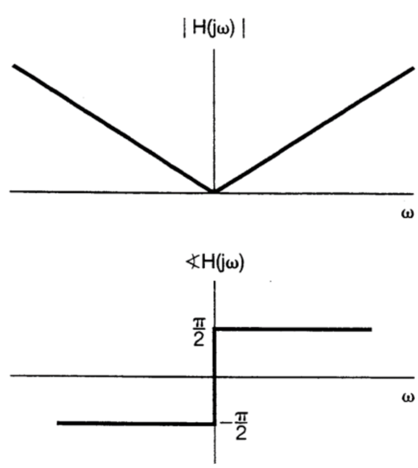


Figure 3.23 Characteristics of the frequency response of a filter for which the output is the derivative of the input.

97

- They are often used to enhance *edges in picture processing* (Fig. 3.24). Because the *derivative* at the *edges* of a picture is *greater* than in regions where the *brightness varies slowly with distance*.

Differentiation on a 2D Image

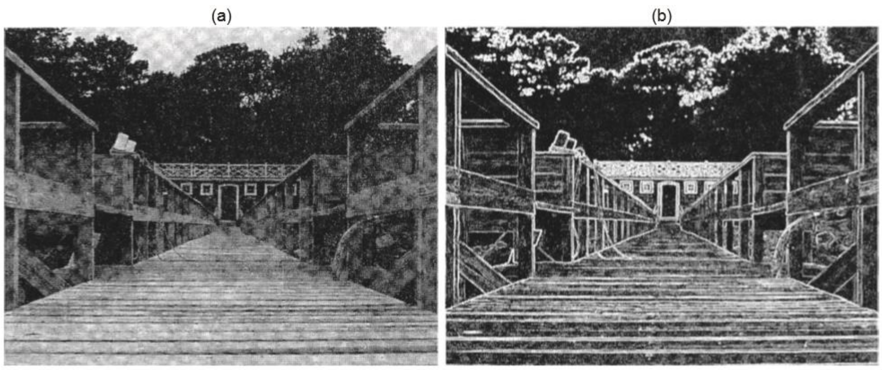


Figure 3.24 Effect of a differentiating filter on an image: (a) the original image; (b) the result of processing the original image with a differentiating filter.

P.261

- A simple discrete-time filter (Fig. 3.25), considering an LTI system that successively take a ***two-point average*** of the input values:

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

$$H(e^{jw}) = \frac{1}{2}(1 + e^{-jw}) = e^{-jw/2}\cos(\frac{w}{2})$$

- In discrete-time we need only consider a 2π interval of values of ω in order to cover a complete range of distinct discrete-time frequencies.
- Fig. 3.25

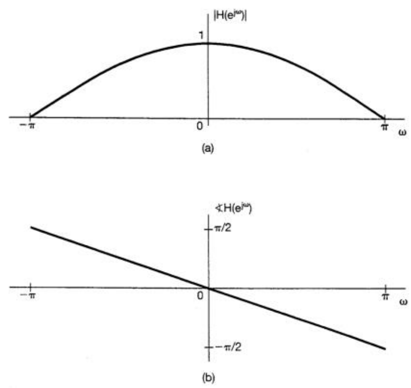


Figure 3.25 (a) Magnitude and (b) phase for the frequency response of the discrete-time LTI system $y[n] = 1/2(x[n] + x[n-1])$.

3.9.2

- Frequency-Selective Filters

1. Low-pass Filter

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases} \qquad (3.140)$$

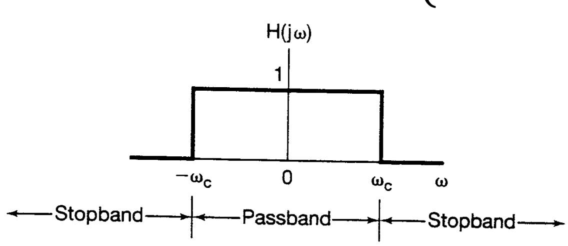


Figure 3.26 Frequency response of an ideal lowpass filter.

3.9.2 Frequency-Selective Filters

2. High-pass Filter & Band-pass Filter

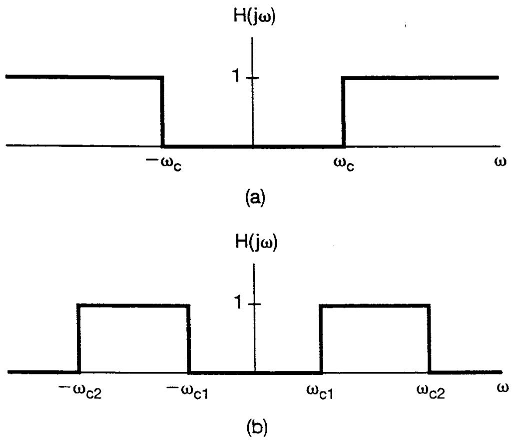


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

102

3.10 Examples of Continuous-Time Filters Described by Differential Equations **(Skip hereafter)**

- 3.10.1 A Simple RC Lowpass Filter

103

3.10.1 A Simple RC Lowpass Filter

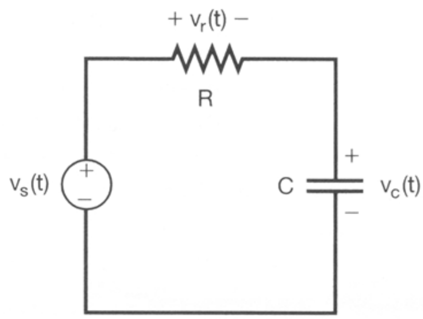


Figure 3.29 First-order RC filter.

Constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t). \qquad (3.141)$$

104

3.10.1 A Simple RC Lowpass Filter

Assuming initial rest, the system described by eq. (3.141) is LTI. In order to determine its frequency response $H(j\omega)$, we note that, by definition, with input voltage $v_s(t) = e^{j\omega t}$, we must have the output voltage $v_c(t) = H(j\omega)e^{j\omega t}$. If we substitute these expressions into eq.(3.141), we obtain

$$RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}, \qquad (3.142)$$

or

$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}, \qquad (3.143)$$

from which it follows directly that

$$H(j\omega)e^{j\omega t} = \frac{1}{1+RCj\omega}e^{j\omega t}, \qquad (3.144)$$

or

$$H(j\omega) = \frac{1}{1+RCj\omega}. \qquad (3.145)$$

105

3.10.1 A Simple RC Lowpass Filter

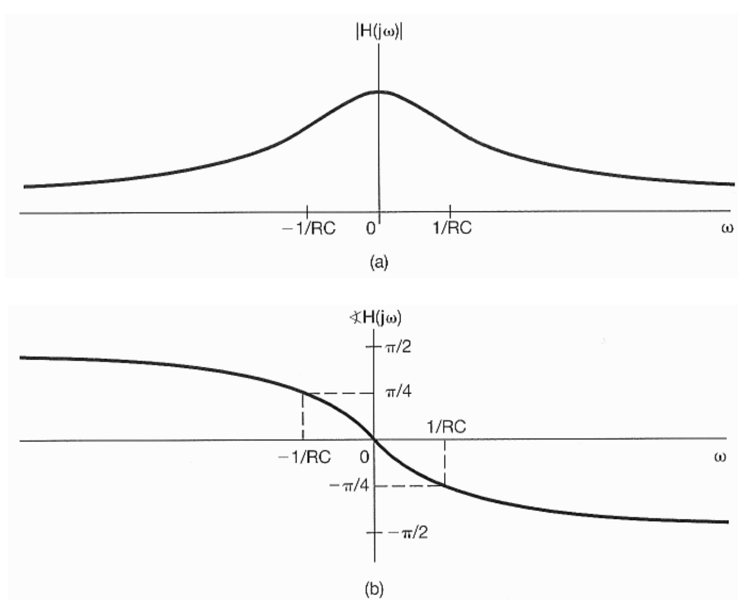
The magnitude and phase of the frequency response $H(j\omega)$ for this example are shown in Figure 3.30. Note that for frequencies near $\omega = 0$, $|H(j\omega)| \approx 1$, while for larger values of ω (positive or negative), $|H(j\omega)|$ is considerably smaller and in fact steadily decreases as $|\omega|$ increases. Thus, this simple RC filter (with $v_c(t)$ as output) is a nonideal lowpass filter.

To provide a first glimpse at the trade-offs involved in filter design, let us briefly consider the time-domain behavior of the circuit. In particular, the impulse response of the system described by eq. (3.141) is

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) \quad (3.146)$$

106

Fig 3.30



107

3.10.1 A Simple RC Lowpass Filter

and the step response is

$$s(t) = \left[1 - e^{-\frac{t}{RC}}\right]u(t), \qquad (3.147)$$

Both of which are plotted in Figure 3.31 (where $\tau = RC$). Comparing Figures 3.30 and 3.31, we see a fundamental trade-off. Specifically, suppose that we would like our filter to pass only very low frequencies. From Figure 3.30(a), this implies that $1/RC$ must be small, or equivalently, that RC is large, so that frequencies other than the low ones of interest will be attenuated sufficiently. However, looking at Figure 3.31(b), we see that if RC is large, then the step response will take a considerable amount of time to reach its long-term value of 1. That is, the system responds sluggishly to the step input. Conversely, if we wish to have a faster step response, we need a smaller value of RC, which in turn implies that the filter will pass higher frequencies. This type of trade-off between behavior in the frequency domain and in the time domain is typical of the issues arising in the design and analysis of LTI systems and filters and is a subject we will look at more carefully in Chapter 6.

108

3.10.1 A Simple RC Lowpass Filter

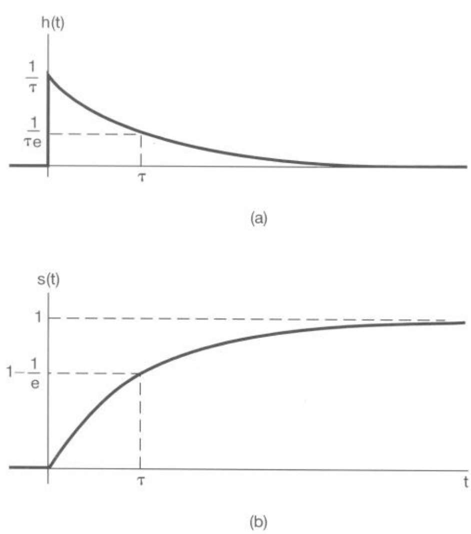


Figure 3.31 (a) Impulse response of the first-order RC lowpass filter with $\tau = RC$; (b) step response of RC lowpass filter with $\tau = RC$.

)9

3.10.2 A Simple RC Highpass Filter

And output is

$$RC\frac{dv_r(t)}{dt} + vr_{(t)} = RC\frac{dv_s(t)}{dt}. \quad (3.148)$$

We can find the frequency response $G(j\omega)$ of this system in exactly the same way we did in the previous case: If $v_s(t) = e^{j\omega t}$, then we must have $v_r(t) = G(j\omega)\, e^{j\omega t}$; substituting these expressions into eq. (3.148) and performing a bit of algebra, we find that

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}. \quad (3.149)$$

The magnitude and phase of this frequency response are shown in Figure 3.32. From the figure, we see that the system attenuates lower frequencies and passes higher frequencies---i.e., those for which |*ω*|>>1/*RC*---with minimal attenuation. That is, this system acts as a nonideal highpass filter.

As with the lowpass filter, the parameters of the circuit control both the frequency response of the highpass filter and its time response characteristics. For example, consider the step response for the filter. From Figure 3.29, we see that $v_r(t) = v_{s(t)} - v_c(t)$. Thus, if $v_s(t)$=u(t), $v_c(t)$ must be given by eq.(3.147). Consequently, the step response of the highpass filter is

$$v_{r(t)} = e^{-\frac{t}{RC}}u(t), \quad (3.150)$$

Which is depicted in Figure 3.33. Consequently, as *RC* is increased, the response becomes more sluggish---i.e., the step response takes a longer time to reach its long-term value 110

3.10.2 A Simple RC Highpass Filter

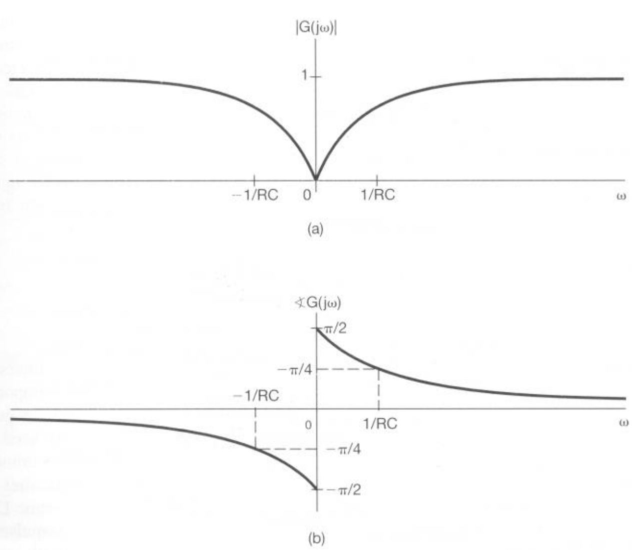


Figure 3.32 (a) Magnitude and (b) phase plots for the frequency response of the *RC* circuit of Figure 3.29 with output $v_r(t)$.

111

3.10.2 A Simple RC Highpass Filter

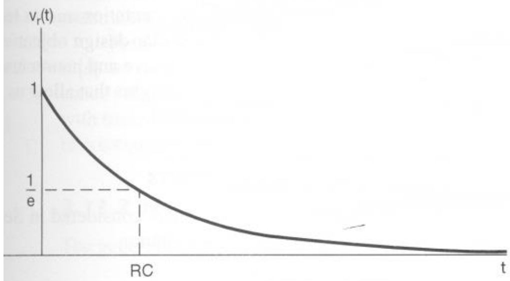


Figure 3.33 Step response of the first-order *RC* highpass filter with $\tau = RC$.

of 0. From Figure 3.32, we see that increasing *RC* (or equivalently, decreasing $1/RC$) also affects the frequency response, specifically, it extends the passband down to lower frequencies.

We observe from the two examples in this section that a simple *RC* circuit can serve as a rough approximation to a highpass or a lowpass filter, depending upon the choice of the physical output variable. As illustrated in Problem 3.71, a simple mechanical system using a mass and a mechanical damper can also serve as a lowpass or highpass filter described by

3.10.2 A Simple RC Highpass Filter

Analogous first-order differential equations. Because of their simplicity, these examples of electrical and mechanical filters do not have a sharp transition from passband to stopband and, in fact, have only a single parameter (namely, *RC* in the electrical case) that controls both the frequency response and time response behavior of the system. By designing more complex filters, implemented using more energy storage elements (capacitances and inductances in electrical filters and springs and damping devices in mechanical filters), we obtain filters described by higher order differential equations. Such filters offer considerably more flexibility in terms of their characteristics, allowing, for example, sharper passband-stopband transition or more control over the trade-offs between time response and frequency response.

3.11 Examples of Discrete-Time Filters Described by Differential Equations

- 3.11.1 First-Order Recursive Discrete-Time Filters

The discrete-time counterpart of each of the first-order filters considered in Section 3.10 is the LTI system described by the first-order difference equation

$$y[n] - ay[n-1] = x[n]. \qquad (3.151)$$

From the eigen function property of complex exponential signals, we know that if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$, where $H(e^{j\omega})$ is the frequency response of the system. Substituting into eq. (3.151), we obtain

$$H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})\, e^{j\omega(n-1)} = e^{j\omega n}, \qquad (3.152)$$

or

$$[1 - ae^{-j\omega}]H(e^{j\omega})\, e^{j\omega n} = e^{j\omega n}, \qquad (3.153)$$

114

3.11.1 First-Order Recursive Discrete-Time Filters

so that

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}. \qquad (3.154)$$

The magnitude and phase of $H(e^{j\omega})$ are shown in Figure 3.34(a) for a=0.6 and in Figure 3.34(b) for a=-0.6. We observe that, for the positive value of a, the difference equation (3.151) behaves like a lowpass filter with minimal attenuation of low frequencies near $\omega = 0$ and increasing attenuation as we increase ω to toward $\omega = \pi$. For the negative value of a, the system is a highpass filter, passing frequencies near $\omega = \pi$ and attenuating lower frequencies. In fact, for any positive value of $a < 1$, the system approximates a lowpass filter, and for any negative value of $a > -1$, the system approximates a highpass filter, where $|a|$ controls the size of the filter passband, with broader passbands as $|a|$ is decreased.

As with the continuous-time examples, we again have a trade-off between time domain and frequency domain characteristics. In particular, the impulse response of the system described by eq. (3.151) is

$$h[n] = a^n u[n]. \qquad (3.155)$$

115

3.11.1 First-Order Recursive Discrete-Time Filters

The step response $s[n] * h[n]$ is

$$s[n] = \frac{1 - a^{n+1}}{1 - a} u[n]. \tag{3.156}$$

From these expressions, we see that $|a|$ also controls the speed with which the impulse and step responses approach their long-term values, with faster responses for smaller values of $|a|$, and hence, for filters with smaller passbands. Just as with differential equations, higher order recursive difference equations can be used to provide sharper filter characteristics and to provide more flexibility in balancing time-domain and frequency-domain constraints.

Finally, note from eq.(3.155) that the system described by eq. (3.151) is unstable if $|a|\geq 1$ and thus does not have a finite response to complex exponential inputs. As we stated previously, Fourier-based methods and frequency domain analysis focus on systems with finite responses to complex exponentials; hence, for examples such as eq.(3.151), we restrict ourselves to stable systems.

116

3.11.1 First-order Recursive Discrete-Time Filters

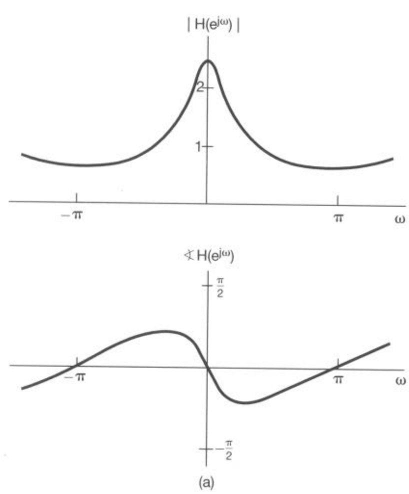


Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a = 0.6$; (b) $a = -0.6$.

117

3.11.1 First-order Recursive Discrete-Time Filters

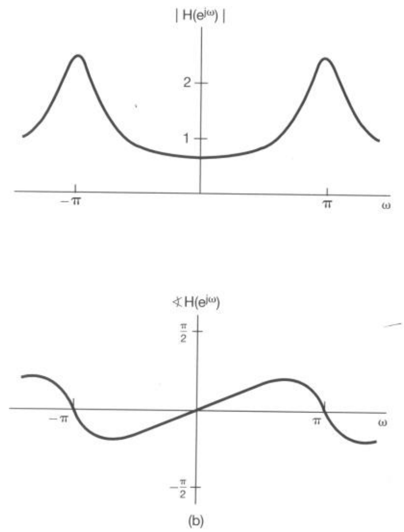


Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a = 0.6$; (b) $a = -0.6$.

118

3.11.2 Nonrecursive Discrete-Time Filters

The general form of an FIR nonrecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \qquad (3.157)$$

That is, the output $y[n]$ is a ***weighted average*** of the $(N + M + 1)$ values of $x[n]$ from $x[n - M]$ through $x[n + N]$, with the weights given by the coefficients b_k. Systems of this form can be used to meet a broad array of filtering objectives, including frequency-selective filtering.

One frequently used example of such a filter is a moving-average filter, where the output $y[n]$ for any n---say, n_0---is an average of values of $x[n]$ in the vicinity of n_0.

119

3.11.2 Nonrecursive Discrete-Time Filters

Basic idea is that by averaging values locally, rapid high-frequency components of the input will be averaged out and lower frequency variations will be retained, corresponding to smoothing or lowpass filtering the original sequence. A simple two-point moving-average filter was briefly introduced in Section 3.9 [eq. (3.138)]. An only slightly more complex example is the three-point moving-average filter, which is of the form

$$y[n] = \frac{1}{3}\left[x[n+1] + x[n] + x[n-1]\right],$$

$$h[n] = \frac{1}{3}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right]$$

And thus, from eq.(3.122), the corresponding frequency response is

$$H\left(e^{j\omega}\right) = \frac{1}{3}\left[e^{j\omega} + 1 + e^{-j\omega}\right] = \frac{1}{3}(1 + 2\cos\omega). \qquad (3.159)$$

The magnitude of $H\left(e^{j\omega}\right)$is sketched in Figure 3.35. We observe that the filter has the general characteristics of a lowpass filter, although, as with the first-order recursive filter, it does not have a sharp transition from passband to stopband.

120

3.11.2 Nonrecursive Discrete-Time Filters

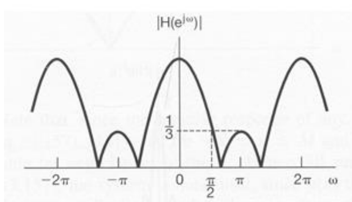


Figure 3.35 Magnitude of the frequency response of a three-point moving-average lowpass filter.

The three-point moving-average filter in eq.(3.158) has no parameters that can be changed to adjust the effective cutoff frequency. As a generalization of this moving-average filter, we can consider averaging over $N + M + 1$ neighboring points---that is, using a difference equation of the form

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]. \qquad (3.160)$$

The corresponding impulse response is a rectangular pulse (i.e., $h[n] = \frac{1}{N+M+1}$ for $-N \le n \le M$, and h[n]=0 otherwise). The filter's frequency response is

$$H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\omega k}. \qquad (3.161)$$

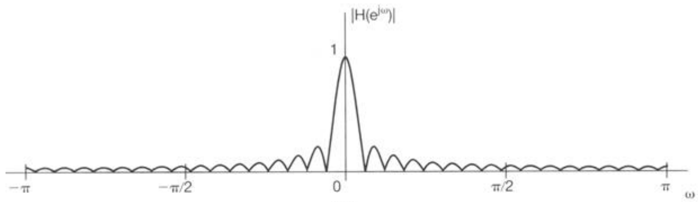
121

3.11.2 Nonrecursive Discrete-Time Filters

The summation in eq.(3.161) can be evaluated by performing calculations similar to those in Example 3.12, yielding

$$H\left(e^{j\omega}\right) = \frac{1}{N+M+1} e^{j\omega k\left[\frac{N-M}{2}\right]} \frac{\sin\left[\frac{\omega(M+N+1)}{2}\right]}{\sin\left(\frac{\omega}{2}\right)}. \qquad (3.162)$$

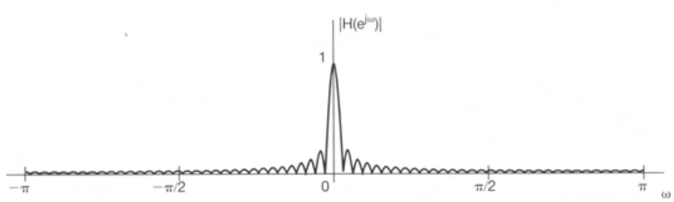
By adjusting the size, $N+M+1$, of the averaging window we can vary the cutoff frequency. For example, the magnitude of $H\left(e^{j\omega}\right)$ is shown in Figure 3.36 for $N+M+1$=33 and $N+M+1=65$.



(a)

122

3.11.2 Nonrecursive Discrete-Time Filters



(b)

Nonrecursive filters can also used to perform highpass filtering operations. To illustrate this, again with a simple example, consider the difference equation

$$y[n] = \frac{x[n]-x[n-1]}{2}. \qquad (3.163)$$

For input signals that are approximately constant, the value of $y[n]$ is close to zero. For input signals that vary greatly from sample to sample, the values of $y[n]$ can be expected to have larger amplitude.

123

3.11.2 Nonrecursive Discrete-Time Filters

Thus, the system described by eq.(3.163) approximates a highpass filtering operation, attenuating slowly varying low-frequency components and passing rapidly varying higher frequency components with little attenuation. To see this more precisely we need to look at the system's frequency response. In this case, $h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$, so that direct application of eq.(3.122) yields

$$H\left(e^{j\omega}\right) = \frac{1}{2}\left[1 - e^{-j\omega}\right] = je^{j\omega/2}\sin\left(\frac{\omega}{2}\right). \qquad (3.164)$$

In Figure 3.37 we have plotted the magnitude of $H\left(e^{j\omega}\right)$, showing that this simple system approximates a highpass filter, albeit one with a very gradual transition from passband to stopband. By considering more general nonrecursive filters, we can achieve far sharper transitions in lowpass, highpass, and other frequency-selective filters.

3.11.2 Nonrecursive Discrete-Time Filters

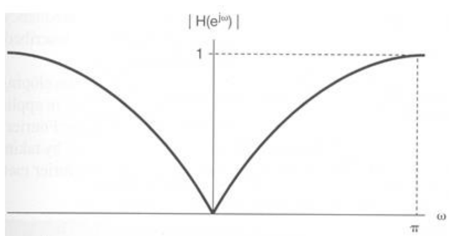


Figure 3.37 Frequency response of a simple highpass filter.

Note that, since the impulse response of any FIR system is of finite length (i.e., from eq.(3.157), $h[n]$=b_n for $-N \le n \le M$ and 0 otherwise), it is always absolutely summable for any choices of the b_n. Hence, all such filters are stable. Also, if N>0 in eq.(3.157), the system is noncausal, since $y[n]$ then depends on future values of the input. In some applications, such as those involving the processing of previously recorded signals, causality is not a necessary constraint, and thus, we are free to use filters with N>0. In others, such as many involving real-time processing, causality is essential, and in such cases we must take $N \le 0$.

Problems

- 3.3, 3.7(3.8 舊版), 3.10, 3.11
- 3.13, 3.14