Chapter 4
The Continuous-Time Fourier Transform

Introduction

- Apply the concepts for periodic signals to ***aperiodic*** (***not periodic***) signals.
- The representation takes ***the form of an integral rather than a sum***.
- The ***resulting spectrum*** of coefficients in this representation is called the ***Fourier transform***.
- The synthesis integral itself, which uses these coefficients to represent the signal as linear combination of complex exponentials, is called the ***inverse Fourier transform***.

4.1 Representation of aperiodic signals: The continuous time Fourier transform

- 4.1.1 Development of Fourier transform representation of an aperiodic signal

Revisit Example 3.5

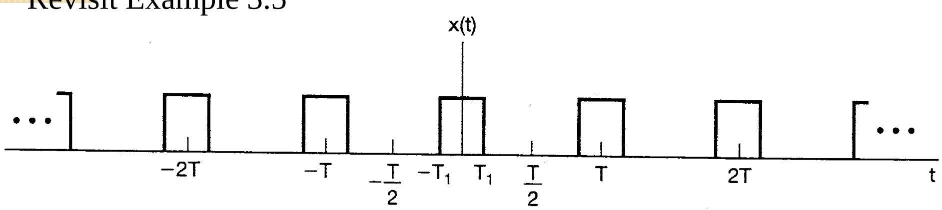

Figure 4.1 A continuous-time periodic square wave.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T_2 \end{cases} \qquad \text{period} = T$$

$$\text{Fourier coefficient } a_k = \frac{2\sin(k\omega_0 T_1)}{kw_0 T}, \ \omega_0 = 2\pi/T$$

$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}\Big|_{\omega = k\omega_0}$$

- With ω thought of as a continuous variable, the function $\frac{2\sin(\omega T_1)}{\omega}$ represents the envelope of Ta_k, and a_k are simply equally spaced sample of this envelope.

- For fixed T_1, the envelope of Ta_k is independent of T. As T increases, or decreases, the envelope is sampled with a closer and closer spacing. $\omega_0 = \frac{2\pi}{T}$

As $T \rightarrow \infty$ original periodic square wave → rectangular pulse, the Fourier coefficients a_k → the envelop function.

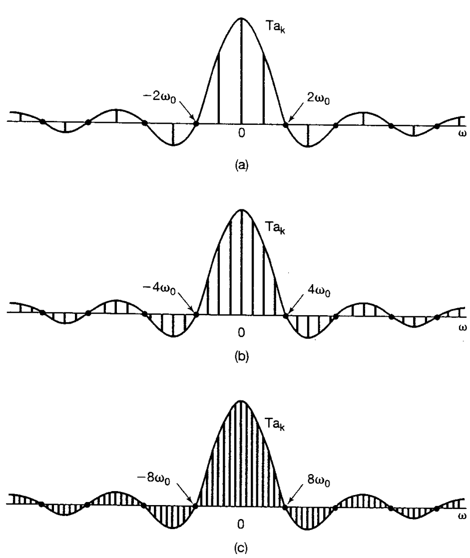

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

- An aperiodic signal $x(t)$ is of finite duration: $x(t)=0$ if $|t|>T_1$

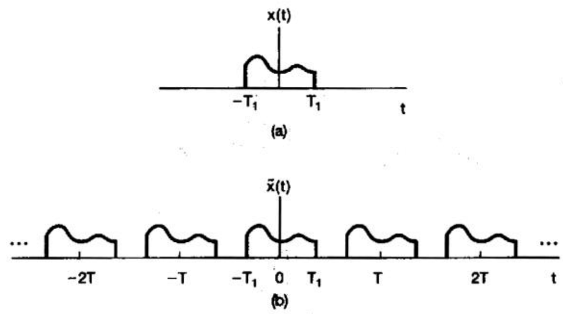

圖 4.3 (a)非週期性信號 $x(t)$；(b)週期性信號 $\tilde{x}(t)$，$x(t)$ 為 $\tilde{x}(t)$ 的一個週期

- We can construct a periodic signal $\tilde{x}(t)$ with period T and for which $x(t)$ is one period.
- Figure 4.3(b)
- As $T\rightarrow\infty$, $\tilde{x}(t) = x(t)$ for any finite value of t

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t} dt \tag{4.3}$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t} dt \tag{4.4}$$

Define the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt, \text{ where } \omega = k\omega_0 \tag{4.5}$$

$$a_k = \frac{1}{T}X(j\omega) = \frac{1}{T}X(jk\omega_0) \tag{4.6}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\omega_0 \quad (\because \omega_0 = \frac{2\pi}{T}) \tag{4.7}$$

As $T \to \infty$, we have $\omega_0 \to 0$ and $\tilde{x}(t) \to x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad (4.8)$$

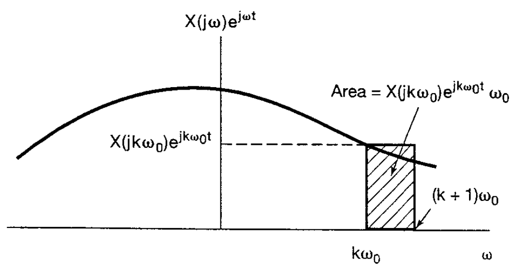

Figure 4.4 Graphical interpretation of eq. (4.7).

Summary of Fourier Transform

Fourier transform pair:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}dt$$ Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$ Inverse Fourier transform

The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the spectrum of $x(t)$

- Let $x(t)$ be a finite-duration signal that is equal to $\tilde{x}(t)$ over exactly one period, $s \leq t \leq s+T$
- The Fourier coefficients a_k of $\tilde{x}(t)$ are proportional to equally spaced samples of the FT of one period of $\tilde{x}(t)$

$$a_k = \frac{1}{T}\int_{s}^{s+T} \tilde{x}(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{s}^{s+T} x(t)\, e^{-jk\omega_0 t} dt$$
$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T} X(j\omega)|_{\omega=k\omega_0}$$

4.1.2 Convergence of Fourier Transform

- Again, the Dirichlet conditions are required.
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4.1.2 Convergence of Fourier Transform

- (1) Square integrable

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \qquad (4.11)$$

- (2) The Dirichlet conditions
  - Condition 1: absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \qquad (4.13)$$

  - Condition 2: finite number of maxima and minima within any finite interval
  - Condition 3: finite number of discontinuities within any finite interval.

4.1.3 Examples of C-T FT

- Examples 4.1~4.5

Example 4.1

$\omega = \pm a$

$\frac{1}{\sqrt{2}\cdot a} = \frac{\sqrt{2}}{2a}$

$z = x + jy, |z| = \sqrt{x^2 + y^2}$
$z = x - jy, |z| = \sqrt{x^2 + (-y)^2}$
$= \sqrt{x^2 + y^2}$
$\angle z = \tan^{-1}\frac{y}{x}$

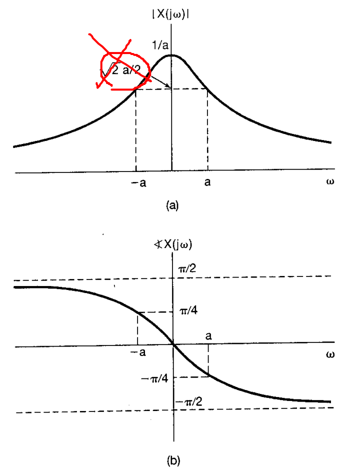


Figure 4.5 Fourier transform of the signal $x(t) = e^{-at}u(t)$, $a > 0$, considered in Example 4.1.

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}, \quad a > 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$\sqrt{2a^2}$

Example 4.2

Let $x(t) = e^{-a|t|}, a > 0$

$\rightarrow X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_0^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

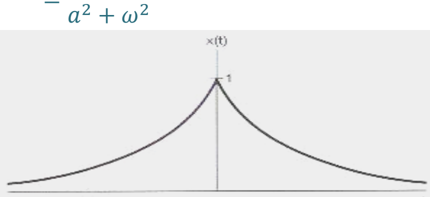


Figure 4.6 Signal $x(t) = e^{-a|t|}$ of Example 4.2.

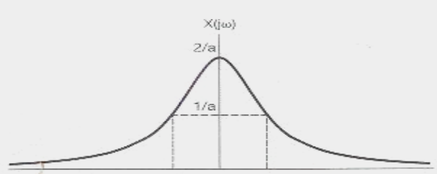


Figure 4.7 Fourier transform of the signal considered in Example 4.2 and depicted in Figure 4.6.

Example 4.3

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t). \tag{4.14}$$

Substituting into eq. (4.9) yields

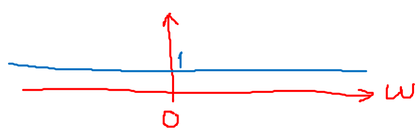
$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1. \tag{4.15}$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at *all* frequencies.

Let $x(t) = \delta(t)$

substituting into Eq. (4.9)

→$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$



Example 4.4

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$$

Applying Eq. (4.9)→$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = 2\frac{\sin \omega T_1}{\omega}$

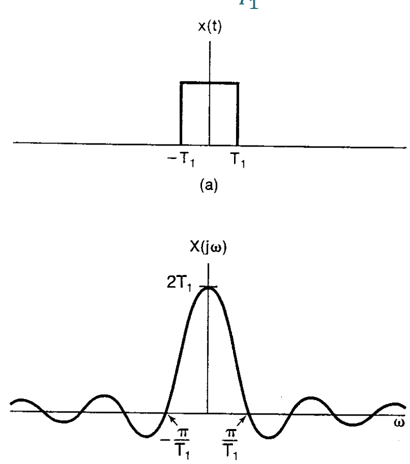


Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

Ex 4.4

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}(e^{-j\omega T_1} - e^{j\omega T_1})$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j}\right)$$

$$= \frac{2}{\omega} \sin \omega T_1$$

Example 4.5

$$X(j\omega) = \begin{cases} 1, |\omega| < W \\ 0, |\omega| < W \end{cases}$$

Using the synthesis equation (4.8), we can determine

→$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t}$

A commonly used precise form for the sinc function is

→$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

$$\frac{2\sin \omega T_1}{\omega} = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$

$T_1 \times \frac{2\sin(\pi \cdot \frac{\omega T_1}{\pi})}{\pi \cdot \frac{\omega T_1}{\pi}}$

$\frac{W}{\pi} \times \frac{\sin(\pi \cdot \frac{Wt}{\pi})}{\pi \cdot \frac{Wt}{\pi}}$

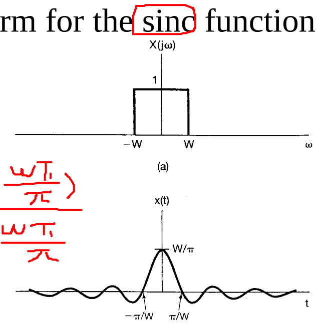


Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

Observation

- The Fourier transform pair consists of a function of the form $\sin a\theta / b\theta$, and a rectangular pulse ➔ ***duality property for Fourier transform***

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Sinc function: $\text{sinc}(\theta) = \dfrac{\sin(\pi\theta)}{\pi\theta}$

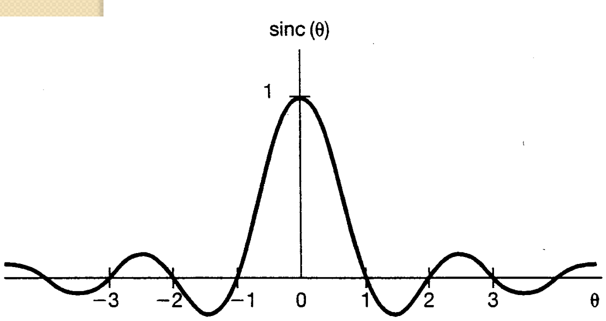


Figure 4.10 The sinc function.

4.2 The Fourier Transform for Periodic Signals

- The FT of a periodic signal can be constructed directly from its FS.
- We consider a signal $x(t)$ with F.T. $X(j\omega)$.

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \qquad (4.21)$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)\, e^{j\omega t} d\omega = e^{j\omega_0 t}$$

More generally, if

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \qquad (4.22)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

F.S. representation of a periodic signal

Example 4.6

Consider again the square wave illustrated in Figure 4.1. The Fourier series coefficients for this signal are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

and the Fourier transform of the signal is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

which is sketched in Figure 4.12 for $T = 4T_1$. In comparison with Figure 3.7(a), the only differences are a proportionality factor of 2π and the use of impulses rather than a bar graph.

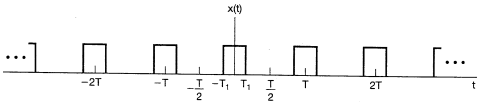


Figure 4.1 A continuous-time periodic square wave.

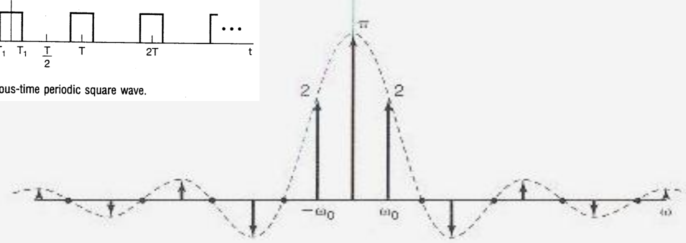


Figure 4.12 Fourier transform of a symmetric periodic square wave.

Example 4.6

- In Figure 4.1 , The Fourier series coefficients for this signal are
  $\rightarrow a_k = \frac{\sin k\omega_0 T_1}{\pi k}$
- And the Fourier transform of the signal is

$$\rightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$

Example 4.7

Let

$$x(t) = \sin \omega_0 t.$$

$= \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$

$k=1$ $k=-1$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \quad k \neq 1 \text{ or } -1.$$

Thus, the Fourier transform is as shown in Figure 4.13(a). Similarly, for

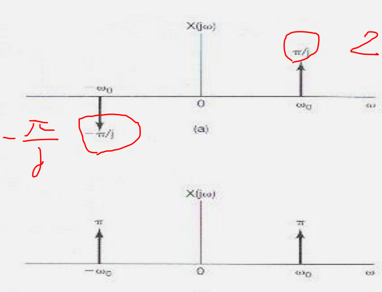
$$x(t) = \cos \omega_0 t,$$

the Fourier series coefficients are

$$a_1 = a_{-1} = \frac{1}{2},$$

$$a_k = 0, \quad k \neq 1 \text{ or } -1.$$

The Fourier transform of this signal is depicted in Figure 4.13(b). These two transforms will be of considerable importance when we analyze sinusoidal modulation systems in Chapter 8.



$2\pi a_1 = 2\pi \times \frac{1}{2j} = \frac{\pi}{j}$

$2\pi a_{-1} = -\frac{\pi}{j}$

Figure 4.13 Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example 4.7

Let $x(t) = \sin \omega_0 t$

Fourier series coefficients for this signal

→ $a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j},$

$a_k = 0$, for $k \neq 1$ or -1

$X(j\omega) = 2\pi a_1 \delta(\omega - \omega_0) + 2\pi a_{-1} \delta(\omega + \omega_0)$

$= \pi/j\, \delta(\omega - \omega_0) - \pi/j\, \delta(\omega + \omega_0)$

Similarly, for $x(t) = \cos \omega_0 t$

→ $a_1 = a_{-1} = \frac{1}{2}$

$a_k = 0$, for $k \neq 1$ or -1

$X(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

Example 4.8

A signal that we will find extremely useful in our analysis of sampling systems in Chapter 7 is the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

which is periodic with period T, as indicated in Figure 4.14(a). The Fourier series coefficients for this signal were computed in Example 3.8 and are given by

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure 4.14(b). Here again, we see an illustration of the inverse relationship between the time and the frequency domains. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

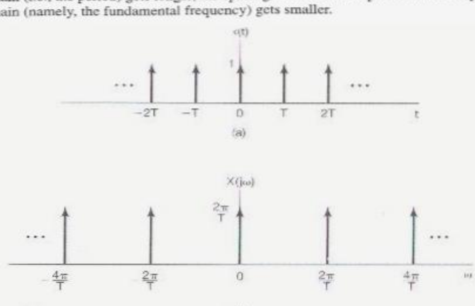


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$.

Example 4.8

Let $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$

which is periodic with period T, as indicated in Figure 4.14. The Fourier series coefficients for this signal were computed in Example 3.8. and are given by

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)e^{jkw_0 t}dt = \frac{1}{T}.$$

Every Fourier coefficient of the periodic impulse train has the same value, $1/T$.
Substituting this value for a_k in Eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

4.3 Properties of the Continuous-Time Fourier Transform

- Linearity:

if

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

and

$$y(t) \overset{F}{\leftrightarrow} Y(j\omega)$$

then

$$ax(t) + by(t) \overset{F}{\leftrightarrow} aX(j\omega) + bY(j\omega) \quad (4.26)$$

- Time shifting:

  if

$$x(t) \overset{F.T.}{\longleftrightarrow} X(j\omega)$$

  then

$$\boxed{x(t-t_0) \overset{F.T.}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)} \quad (4.27)$$

- Conjugation and Conjugate Symmetry:

  if

$$x(t) \overset{F.T.}{\longleftrightarrow} X(j\omega)$$

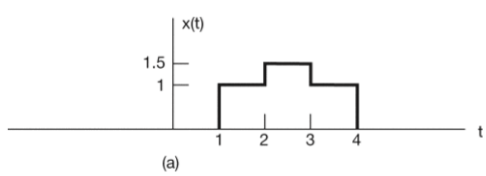
  then

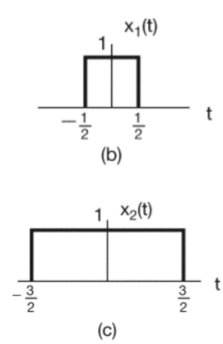
$$\boxed{x^*(t) \overset{F.T.}{\longleftrightarrow} X^*(-j\omega)} \quad (4.28)$$

if $x(t)$ is real, then $X(j\omega)$ is symmetric; that is:

$$X(-j\omega) = X^*(j\omega) \quad (4.30)$$

[Example 4.9] $x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$





From [Example 4.4]

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \qquad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

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- If $x(t)$ is real and we express $X(j\omega)$ in rectangular form as:

$$X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\mathrm{Im}\{X(j\omega)\}$$

$$\Rightarrow \mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$

- If $x(t)$ is both *real and even*, then $X(j\omega)$ is also *real and even*.
- If $x(t)$ is *real and odd*, then $X(j\omega)$ is purely *imaginary and odd*.

Example 4.10

Consider again the Fourier transform evaluation of Example 4.2 for the signal $x(t) = e^{-a|t|}$, where $a > 0$. This time we will utilize the symmetry properties of the Fourier transform to aid the evaluation process.

From Example 4.1, we have

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + jw}$$

Note that for $t > 0$, $x(t)$ equals $e^{-at}u(t)$, while for $t < 0$, $x(t)$ takes on mirror image values. That is,

$$\begin{aligned} x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathcal{E}v\{e^{-at}u(t)\}. \end{aligned}$$

Since $e^{-at}u(t)$ is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathcal{E}v\{e^{-at}u(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\}.$$

It follows that

$$X(j\omega) = 2\mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2},$$

$$x(t) \overset{f.t.}{\longleftrightarrow} X(jw)$$

$$Ev\{x(t)\} \overset{f.t.}{\longleftrightarrow} \mathrm{Re}\{X(jw)\}$$

which is the same as the answer found in Example 4.2.

Example 4.10

From Example 4.1, we have

→ $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$

Note that for $t>0$, $x(t)$ equals $e^{-at}u(t)$, while for $t<0$, $x(t)$ takes on mirror image values. That is

→ $x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2\mathcal{E}v\{e^{-at}u(t)\}$$

Since $e^{-at}u(t)$ is real valued, the symmetry properties of the Fourier transform lead us to conclude that

→ $\mathcal{E}v\{e^{-at}u(t)\} \overset{\mathcal{F}}{\longleftrightarrow} \Re e\left\{\frac{1}{a+j\omega}\right\}$

It follow that $X(j\omega) = 2\Re e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$

- Differentiation and Integration:

  if

$$x(t) \overset{F.T.}{\longleftrightarrow} X(j\omega)$$

  then

$$\frac{d}{dt}x(t) \overset{F.T.}{\longleftrightarrow} j\omega X(j\omega) \quad (4.31)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F.T.}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \quad (4.32)$$

- Time and Frequency Scaling:

  if

$$x(t) \overset{F.T.}{\longleftrightarrow} X(j\omega)$$

  then

$$x(at) \overset{F.T.}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \quad (4.34)$$

Example 4.11

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t) = u(t)$, making use of eq. (4.32) and the knowledge that

$$g(t) = \delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

and taking the Fourier transform of both sides, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

where we have used the integration property listed in Table 4.1. Since $G(j\omega) = 1$, we conclude that

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \qquad (4.33)$$

Observe that we can apply the differentiation property of eq. (4.31) to recover the transform of the impulse. That is,

$$\delta(t) = \frac{du(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

where the last equality follows from the fact that $\omega\delta(\omega) = 0$.

$j\omega\pi\delta(\omega)$

$\delta(\omega) = \begin{cases} 1, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases}$

$= j\cdot 0\cdot\pi\delta(\omega)$

$= 0$

Example 4.11

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t)=u(t)$, making use of eq.(4.32) and the knowledge

$$g(t) = \delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} G(j\omega) = 1$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

And taking the Fourier transform of both sides, we obtain

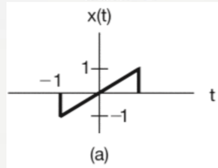
$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

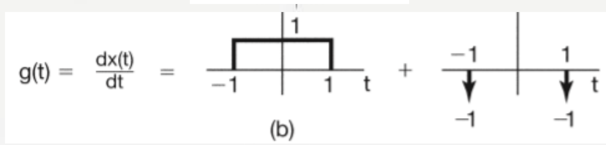
Since $G(j\omega)=1$ → $X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$ ---- −(4.33)

Apply Eq. (1.31) → $\delta(t) = \frac{du(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$=1

Example 4.12

Suppose that we wish to calculate the Fourier transform $X(j\omega)$ for the signal $x(t)$ displayed in Figure 4.16(a). Rather than applying the Fourier integral directly to $x(t)$, we instead consider the signal





As illustrated in Figure 4.16(b), $g(t)$ is the sum of a rectangular pulse and two impulses. The Fourier transforms of each of these component signals may be determined from Table 4.2:

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}.$$

Note that $G(0) = 0$. Using the integration property, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

With $G(0) = 0$ this becomes

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}.$$

The expression for $X(j\omega)$ is purely imaginary and odd, which is consistent with the fact that $x(t)$ is real and odd.

Example 4.12

As illustrated in Figure 4.16(b), $g(t)$ is the sum of a rectangular pulse and two impulses.

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

Note that $G(0)$=0. Using the integration property, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

With $G(0)$=0, this becomes

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2}{j\omega}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

4.3.5 Time and Frequency Scaling

If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$. $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$

then since $\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt$

after substituting at by τ

$$\mathcal{F}\{x(at)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ -\frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases}$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$$

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$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$$

Specially, when a = 1,

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

(time reversal property)

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4.3.6 Duality

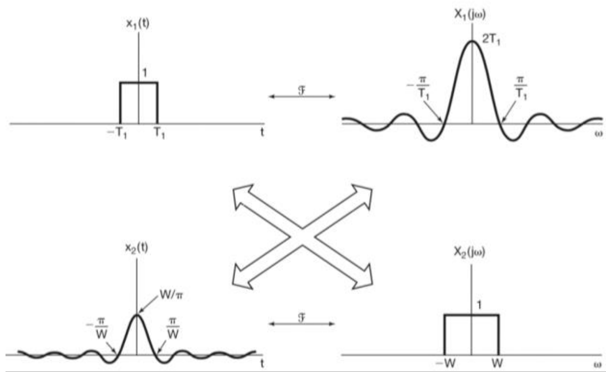
(1) Duality for Transform Pairs

In summary, if $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$X(jt) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

$$\frac{1}{2\pi} X(jt) \stackrel{\mathcal{F}}{\longleftrightarrow} x(-\omega)$$

P.337-338



Duality

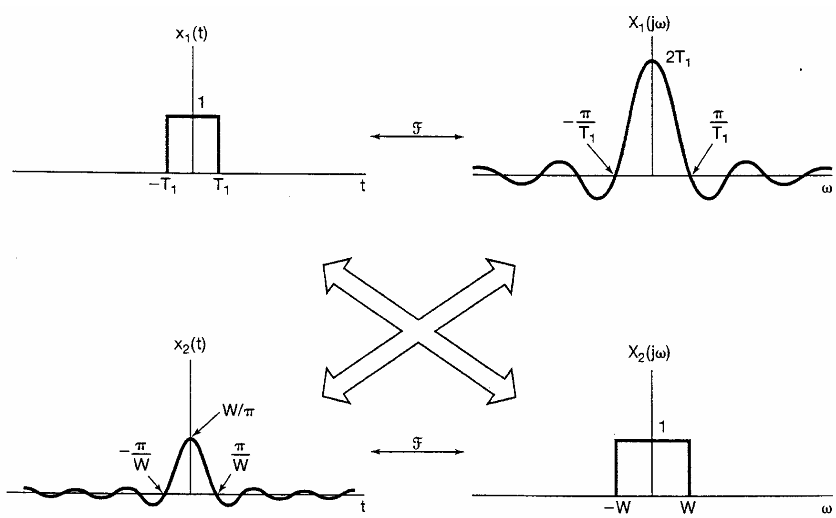


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

Example 4.13

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

In Example 4.2 we encountered a Fourier transform pair in which the Fourier transform, as a function of ω, had a form similar to that of the signal $x(t)$. Specifically, suppose we consider a signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

Then, from Example 4.2,

$$e^{-a|t|} \xrightarrow{a=1} X(j\omega) = \frac{2}{a^2+\omega^2}$$

$$x(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

Now, interchanging the names of the variables t and ω, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt. \quad (4.38)$$

The right-hand side of eq. (4.38) is the Fourier transform analysis equation for $2/(1+t^2)$, and thus, we conclude that

$$\mathcal{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}.$$

Example 4.13

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}$$

In Example 4.2

$\rightarrow X(j\omega) = \frac{2}{1+\omega^2}$

$\rightarrow x(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

$t \rightarrow -t \quad \rightarrow \quad 2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega$

$t \rightarrow \omega \quad \rightarrow \quad 2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt$

$\rightarrow \mathcal{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$

- If $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

then

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}dt \qquad (4.39)$$

that is:

$$-jtx(t) \overset{FT}{\leftrightarrow} \frac{dX(j\omega)}{d\omega} \qquad (4.40)$$

similarly:

$$e^{j\omega_0 t}x(t) \overset{FT}{\leftrightarrow} X(j(\omega - \omega_0)) \qquad (4.41)$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{FT}{\leftrightarrow} \int_{-\infty}^{W} X(\eta)d\eta \qquad (4.42)$$

4.3.7 Parseval's Relation

- Parseval's Relation:

If

$$x(t) \overset{F.T.}{\longleftrightarrow} X(j\omega)$$

then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \qquad (4.43)$$

(Proof)

$$\int_{-\infty}^{+\infty}|x(t)|^2\,dt=\int_{-\infty}^{+\infty}x(t)x^*(t)dt$$

$$=\int_{-\infty}^{+\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)e^{-j\omega t}d\omega\right]dt.$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)\left[\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\right]d\omega.$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2\,d\omega$$

Example 4.14

For each of the Fourier transforms shown in Figure 4.18, we wish to evaluate the following time-domain expressions:

$$E=\int_{-\infty}^{\infty}|x(t)|^2dt$$

$$D=\frac{d}{dt}x(t)\Big|_{t=0}$$

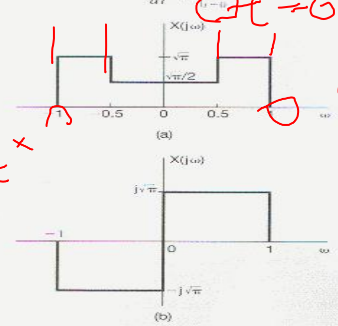


Figure 4.18 The Fourier transforms considered in Example 4.14.

To evaluate E in the frequency domain, we may use Parseval's relation. That is,

$$E=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega \qquad (4.44)$$

which evaluates to $\frac{5}{8}$ for Figure 4.18(a) and to 1 for Figure 4.18(b).

To evaluate D in the frequency domain, we first use the differentiation property to observe that

$$g(t)=\frac{d}{dt}x(t)\overset{\mathcal{F}}{\longleftrightarrow}j\omega X(j\omega)=G(j\omega).$$

Noting that

$$D=g(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(j\omega)d\omega \qquad (4.45)$$

we conclude:

Example 4.14

For each of the Fourier transforms shown in Fig. 4.18, we wish to evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt\,, \qquad D = \frac{d}{dt}x(t)\bigg|_{t=0}$$

To evaluate E, we may use Parseval's relation. That is

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \ \ldots\ldots\ldots\ldots\ldots (4.44)$$

To evaluate *D* in the frequency domain, we first use the differentiation property to observe that

$$g(t) = \frac{d}{dt}x(t) \overset{\mathcal{FT}}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega)$$

Noting that→ $D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega \ \ldots\ldots\ldots\ldots\ldots (4.45)$

We conclude→ $D = \int_{-\infty}^{\infty} j\omega X(j\omega)d\omega \ \ldots\ldots\ldots\ldots\ldots\ldots (4.46)$

Fig. 4.18(a)

$$E = \frac{1}{2\pi}\int_{-1}^{1} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\left\{ \int_{-1}^{-0.5} |\sqrt{\pi}|^2 d\omega + \int_{-0.5}^{0.5} \left|\frac{\sqrt{\pi}}{2}\right|^2 d\omega + \int_{0.5}^{1} |\sqrt{\pi}|^2 d\omega \right\}$$

$$= \frac{1}{2\pi}\left\{ \pi\omega\Big|_{-1}^{-0.5} + \frac{\pi}{4}\omega\Big|_{-0.5}^{0.5} + \pi\omega\Big|_{0.5}^{1} \right\}$$

$$= \frac{1}{2\pi}\left\{ 0.5\pi + \frac{\pi}{4} + \frac{\pi}{2} \right\}$$

$$= \frac{1}{2\pi} \times \frac{5\pi}{4} = \frac{5}{8} \ ※$$

Figure 4.18(a)

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)\, d\omega$$

$$= \frac{j}{2\pi}\left\{\int_{-1}^{-0.5}\sqrt{\pi}\,\omega\, d\omega + \int_{-0.5}^{0.5}\frac{\sqrt{\pi}}{2}\omega\, d\omega + \int_{0.5}^{1}\sqrt{\pi}\,\omega\, d\omega\right\}$$

$$= \frac{j\sqrt{\pi}}{2\pi}\left\{\frac{1}{2}\omega^2\Big|_{-1}^{-0.5} + \frac{1}{4}\omega^2\Big|_{-0.5}^{0.5} + \frac{1}{2}\omega^2\Big|_{0.5}^{1}\right\}$$

$$= \frac{j}{2\sqrt{\pi}}\left\{\frac{-0.75}{2} + 0 + \frac{0.75}{2}\right\}$$

$$= 0$$

Fig. 4.18(b)

$$E = \frac{1}{2\pi}\int |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\left\{\int_{-1}^{0}|-j\sqrt{\pi}|^2 d\omega + \int_{0}^{1}|j\sqrt{\pi}|^2 d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\pi\omega\Big|_{-1}^{0} + \pi\omega\Big|_{0}^{1}\right\}$$

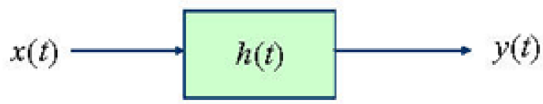
$$= \frac{1}{2\pi}\{\pi + \pi\} = \frac{2\pi}{2\pi} = 1 \quad \#$$

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)\, d\omega$$

$$= \frac{j}{2\pi}\left\{\int_{-1}^{0} -j\sqrt{\pi}\,\omega\, d\omega + \int_{0}^{1} j\sqrt{\pi}\,\omega\, d\omega\right\} = \cdots = \frac{-1}{2\sqrt{\pi}} \quad \#$$

4.4 The Convolution Property

- For an LTI system:



$$=> y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(j\omega) = F\{y(t)\}$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \qquad (4.53)$$
$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau \qquad (4.54)$$

$e^{-j\omega\tau}H(j\omega)$

$X(j\omega)$

$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}d\tau \qquad (4.55)$$
$$= H(j\omega)\, X(j\omega)$$

- Summary:

$$y(t) = x(t) * h(t) \overset{F.T.}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega) \qquad (4.56)$$

Cascade of two LTI systems:

$$Y(j\omega) = Y_1(j\omega)H_2(j\omega) = H_1(j\omega)H_2(j\omega)X(j\omega)$$

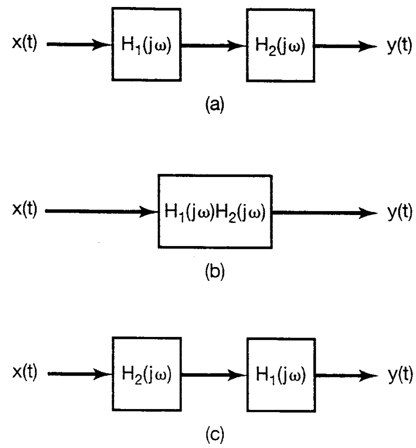


Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

Examples 4.15~4.20

Example 4.15

- Consider a continuous-time LTI system with impulse response
  $\rightarrow h(t) = \delta(t - t_0) \ldots\ldots\ldots\ldots\ldots\ldots (4.58)$
- The Fourier transform of $h(t)$
  $\rightarrow H(j\omega) = e^{-j\omega t_0} \ldots\ldots\ldots\ldots\ldots\ldots (4.59)$
- For any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

$$\rightarrow Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega) \ldots (4.60)$$

- This result, in fact, is consistent with the time-shift property of Section 4.3.2. Specifically, a system for which the impulse response is $\delta(t - t_0)$ applies a time shift to the input —
- that is $\rightarrow y(t) = x(t - t_0)$

Example 4.16

As a second example, let us examine a differentiator—that is, an LTI system for which the input $x(t)$ and the output $y(t)$ are related by

$$\rightarrow y(t) = \frac{dx(t)}{dt}$$

From the differentiation property of Section 4.3.4

$$\rightarrow Y(j\omega) = j\omega X(j\omega) \ldots\ldots\ldots\ldots\ldots (4.61)$$

Consequently, from Eq. (4.56), it follows that the frequency response of a differentiator is

$$\rightarrow H(j\omega) = j\omega$$

Example 4.17

Consider an integrator—that is, an LTI system specified by the equation

→$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ ← = ?
$\pi X(j\omega)\delta(\omega)$ ←

The impulse response for this system is the unit step u(t), and therefore, from Example 4.11 and Eq. (4.33), the frequency response of the system is

→$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

Then using eq. (4.56), we have

→$Y(j\omega) = H(j\omega)X(j\omega)$

$= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega)$

$= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

which is consistent with the integration property of Eq. (4.32).

Example 4.18

- As we discussed in Section 3.9.2, frequency-selective filtering is accomplished with an LTI system whose frequency response $H(j\omega)$ passes the desired range of frequencies and significantly attenuates frequencies outside that range.
- For example, consider the ideal lowpass filter introduced in Section 3.9.2, which has the frequency response illustrated in Figure 4.20 and given by

  →$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$..................(4.63)
- Now that we have developed the Fourier transform representation, we know that the impulse response $h(t)$ of this ideal filter is the inverse transform of eq. (4.63). Using the result in Example 4.5, we then have

  →$h(t) = \frac{\sin \omega_c t}{\pi t}$ … … … … … (4.64)

  which is plotted in Figure 4.21.

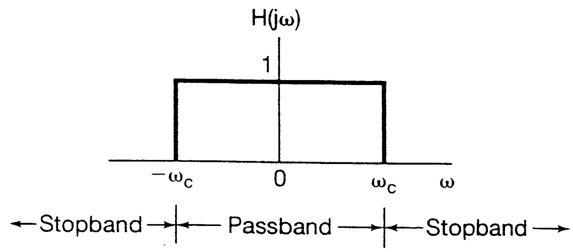


Figure 4.20 Frequency response of an ideal lowpass filter.

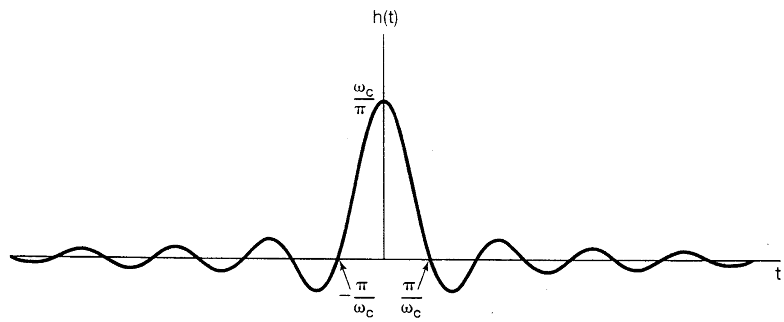


Figure 4.21 Impulse response of an ideal lowpass filter.

- From Example 4.18, we can begin to see some of the issues, that arise in filter design that involve looking in both the time and frequency domains.
- In particular, while the ideal lowpass filter does have perfect frequency selectivity, its impulse response has some characteristics that may not be desirable.
- First, note that $h(t)$ is not zero for $t < 0$. Consequently, the ideal lowpass filter is not causal, and thus, in applications requiring causal systems, the ideal filter is not an option.
- Moreover, as we discuss in Chapter 6, even if causality is not an essential constraint, the ideal filter is not easy to approximate closely, and non-ideal filters that are more easily implemented are typically preferred.
- Furthermore, in some applications (such as the automobile suspension system discussed in Section 6.7.1), oscillatory behavior in the impulse response of a lowpass filter may be undesirable.

- In such applications the time domain characteristics of the ideal lowpass filter, as shown in Figure 4.21, may be unacceptable, implying that we may need to trade off frequency-domain characteristics such as ideal frequency selectivity with time-domain properties.
- For example, consider the LTI system with impulse response

$$\rightarrow h(t) = e^{-t}u(t) \dots \dots \dots \dots \dots (4.65)$$

- The frequency response of this system is

$$\rightarrow H(j\omega) = \frac{1}{j\omega+1} \dots \dots \dots (4.66)$$

Fig. 4.22

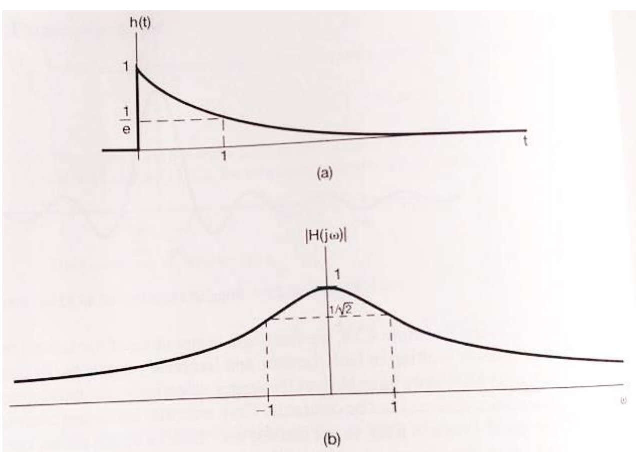


Figure 4.22 (a) Impulse response of the LTI system in eq. (4.65); (b) magnitude of the frequency response of the system.

Example 4.19

Consider the response of an LTI system with impulse response

→$h(t) = e^{-at}u(t), \quad a > 0$

To the input signal

→$x(t) = e^{-bt}u(t), \quad b > 0$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain. From Example 4.1, the Fourier transforms of $x(t)$ and $h(t)$ are→$X(j\omega) = \frac{1}{b+j\omega}, \quad H(j\omega) = \frac{1}{a+j\omega}$

Therefore, → $Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$ … … … … … (4.67)

- To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a ***partial-fraction expansion***.
  - Such expansions are extremely useful in evaluating inverse transforms
  - The general method for performing a partial-fraction expansion is developed in the appendix.
- For this example, **assuming that $b \neq a$**, the partial fraction expansion for $Y(j\omega)$ takes the form

→$Y(j\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}$ ……………………….(4.68)

Where A and B are constants to be determined. One way to find A and B is to equate the right-hand sides of Eqs. (4.67) and (4.68), multiply both sides by $(a+j\omega)(b+j\omega)$, and solve for A and B.

$1=A(b+j\omega)+B(a+j\omega)$,

$Ab+Ba=1,\ j\omega(A+B)=0$ ➔ $B=-A,\ A(b-a)=1$,

We find that→$A=\frac{1}{b-a}=-B$

$$\therefore Y(j\omega)=\frac{1}{b-a}\left[\frac{1}{a+j\omega}-\frac{1}{b+j\omega}\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots (4.69)$$

The inverse transform for each of the two terms in Eq. (4.69) can be recognized by inspection. Using the linearity property of Section 4.3.1, we have

$\rightarrow y(t)=\frac{1}{b-a}\left[e^{-at}u(t)-e^{-bt}u(t)\right]$

When $b=a$, the partial fraction expansion of Eq. (4.69) is not valid. However, with $b=a$, Eq. (4.67) becomes

$\rightarrow Y(j\omega)=\frac{1}{(a+j\omega)^2}=j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$

We can use the dual of the differentiation property, as given in Eq. (4.40). Thus,

$$\rightarrow \begin{cases} e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{a+j\omega} \\ te^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]=\frac{1}{(a+j\omega)^2} \end{cases} \Rightarrow y(t)=te^{-at}u(t)$$

Example 4.20

As another illustration of the usefulness of the ***convolution property***, let us consider the problem of determining the response of an ideal lowpass filter to an input signal $x(t)$ that has the form of a sine function. That is

$$\rightarrow x(t) = \frac{\sin \omega_i t}{\pi t}$$

Of course, the impulse response of the ideal lowpass filter is of a similar form, namely,

$$\rightarrow h(t) = \frac{\sin \omega_c t}{\pi t}$$

The filter output $y(t)$ will therefore be the convolution of two sinc functions, which, as we now show, also turns out to be a sinc function.

A particularly convenient way of deriving this result is to first observe that

$$\rightarrow Y(j\omega) = X(j\omega)H(j\omega)$$

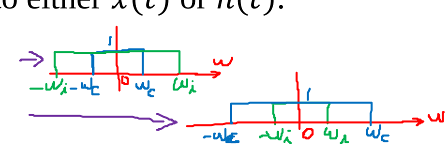
$$\left(X(j\omega) = \begin{cases} 1, & |\omega| \le \omega_i \\ 0, & \text{elsewhere} \end{cases}, \quad H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{elsewhere} \end{cases} \right)$$

$$\rightarrow Y(j\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & \text{elsewhere} \end{cases}$$

where ω_0 is the smaller of the two numbers ω_c and ω_i. Finally, the inverse Fourier transform of $Y(j\omega)$ is given by

$$\rightarrow y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t}, & \text{if } \omega_c \le \omega_i \\ \dfrac{\sin \omega_i t}{\pi t}, & \text{if } \omega_i \le \omega_c \end{cases}$$

That is, depending upon which of ω_c and ω_i is smaller, the output is equal to either $x(t)$ or $h(t)$.



4.5 The Multiplication Property

$$r(t) = s(t)p(t) \overset{F.T.}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)] \qquad (4.70)$$

$$r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Example 4.21

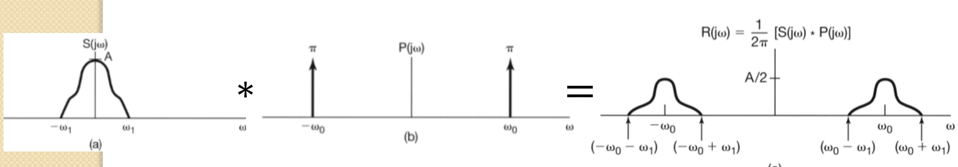
Determine the F. T. of $r(t) = s(t)p(t)$, where $s(t) \overset{F.T.}{\longleftrightarrow} S(j\omega)$ and $p(t) = cos\omega_0 t$.

$$p(t) = cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

$$= \frac{1}{2}S\big(j(\omega - \omega_0)\big) + \frac{1}{2}S\big(j(\omega + \omega_0)\big)$$

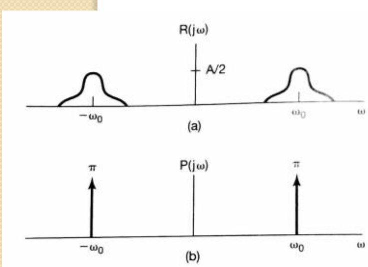


Example 4.22

- Let us now consider $r(t)$ as obtained in Example 4.21, and let
  →$g(t) = r(t)p(t)$

where, again, $p(t) = \cos\omega_0 t$. Then, $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are as shown in Figure 4.24.

- From Figure 4.24(c) and the linearity of the Fourier transform, we see that $g(t)$ is the sum of $(1/2)s(t)$ and a signal with a spectrum that is nonzero only at higher frequencies (centered around :$\pm 2\omega_0$).



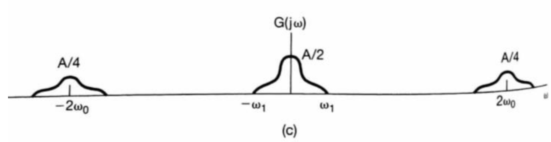


Figure 4.24 Spectra of signals considered in Example 4.22: (a) $R(j\omega)$; (b) $P(j\omega)$; (c) $G(j\omega)$.

- Suppose then that we apply the signal $g(t)$ as the input to a ***frequency-selective lowpass*** filter with frequency response $H(j\omega)$ that is constant at low frequencies (say, for $|\omega| < \omega_1$) and zero at high frequencies (for $|\omega| > \omega_1$).

- Then the output of this system will have as its spectrum $H(j\omega)G(j\omega)$, which, because of the particular choice of $H(j\omega)$, will be a ***scaled replica*** of $S(j\omega)$. Therefore, the output itself will be a scaled version of $s(t)$.

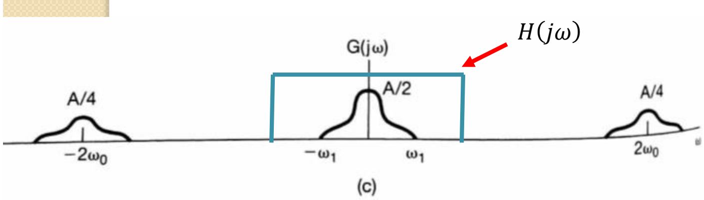
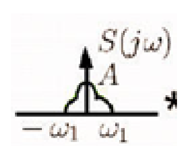


Figure 4.24 Spectra of signals considered in Example 4.22: (a) $R(j\omega)$; (b) $P(j\omega)$; (c) $G(j\omega)$.

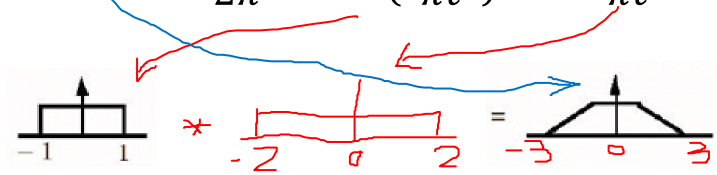


Example 4.23

- What is the F. T. of $x(t) = \dfrac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$?

$$x(t) = \pi\left(\frac{\sin t}{\pi t}\right)\left(\frac{\sin\frac{t}{2}}{\pi t}\right)$$

$$X(j\omega) = \frac{1}{2\pi}\cdot\pi\cdot F\left\{\frac{\sin t}{\pi t}\right\} * F\left\{\frac{\sin\frac{t}{2}}{\pi t}\right\}$$



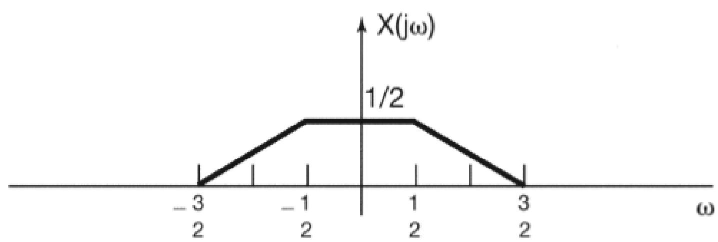


Figure 4.25 The Fourier transform of $x(t)$ in Example 4.23.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

- Figure 4.26

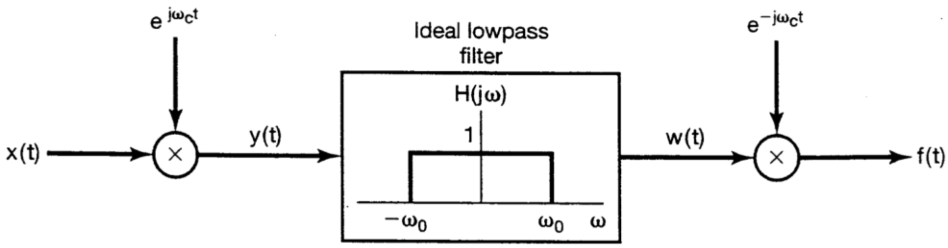


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

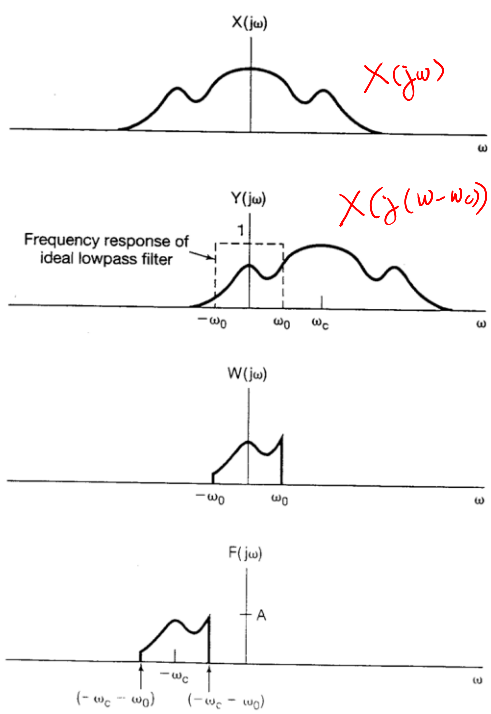


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

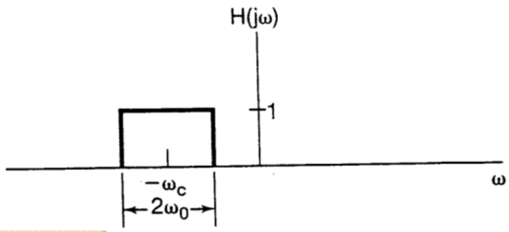


Figure 4.28 Bandpass filter equivalent of Figure 4.26.

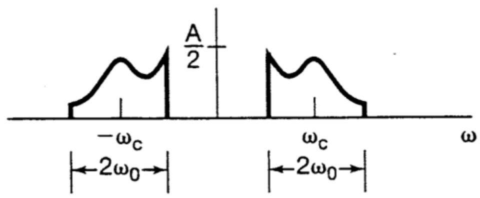


Figure 4.29 Spectrum of $\mathcal{R}e\{f(t)\}$ associated with Figure 4.26.

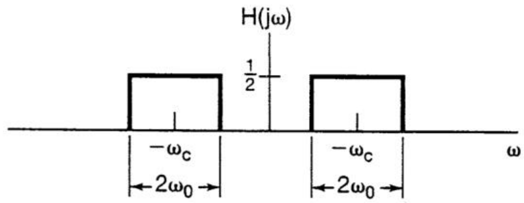


Figure 4.30 Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$ in Figure 4.29.

4.6 Tables of Fourier Properties and of Basic Fourier Transform Pairs

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$

(followed by the next page)

$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

- Linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (4.72)$$

If $Y(j\omega) = H(j\omega)X(j\omega)$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad (4.73)$$

and then:

$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \qquad (4.74)$$

$$\Rightarrow \sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\} \qquad (4.75)$$

$$\Rightarrow \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \qquad (4.76)$$

Example 4.24

Consider a stable LTI system characterized by the differential equation

$j\omega Y(j\omega) + aY(j\omega) = X(j\omega) = (j\omega + a)Y(j\omega)$

$$\frac{dy(t)}{dt} + ay(t) = x(t). \qquad (4.77)$$

with $a>0$. From Eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \qquad (4.78)$$

Comparing this with the result of Example 4.1, we see that Eq. (4.78) is the Fourier transform of $e^{-at}u(t)$.

The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example 4.25

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Taking $\mathcal{F}\{\ \}$, one obtains

$$Y(j\omega)[(j\omega)^2 + 4(j\omega) + 3] = X(j\omega)(j\omega + 2)$$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

Impulse response

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Example 4.26

- If $x(t) = e^{-t}u(t)$, $\rightarrow X(j\omega) = \frac{1}{j\omega+1}$

$Y(j\omega) = H(j\omega)X(j\omega)$

$$= \left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right] = \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

- The partial fractional expansion takes the form

$$y(j\omega) = \frac{A_{11}}{j\omega+1} + \frac{A_{12}}{(j\omega+1)^2} + \frac{A_{21}}{j\omega+3},$$

where A_{11}, A_{12}, and A_{21} are constants to be determined.

$$A_{11}=1/4,\ A_{12}=1/2, \text{ and } A_{21}=1/4$$

With the inverse Fourier transform,

$$y(t) = [\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t}]u(t).$$

Homework

- Basic problems:
- 4.1, 4.3, 4.4, 4.10, 4.12, 4.13, 4.14, 4.15
- 4.33, 4.37(a), (b), 4.38