Chapter 5

The Discrete-Time Fourier Transform

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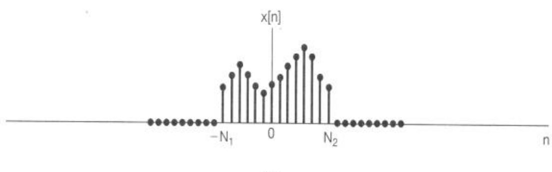
5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Recall that in the aperiodic square wave example in 4.1

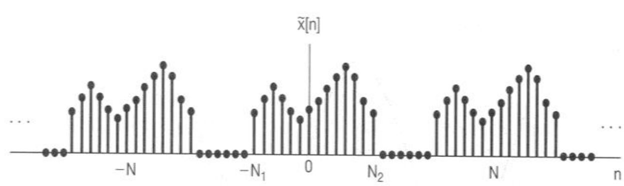
$$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega=k\omega_0}$$

- With $T\rightarrow\infty$, Ta_k approaches the envelope
- Consider an aperiodic sequence $x[n]$ which is of finite duration



Finite–duration signal $x[n]$, $x[n]$=0 outside $-N_1 \le n \le N_2$

- Construct a periodic sequence $\tilde{x}[n]$
- Observe that $\lim\limits_{N\to\infty} \tilde{x}[n] = x[n]$
- Since $\tilde{x}[n]$ is periodic,



$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n} \qquad (5.1)$$

$$a_k = \frac{1}{N}\sum_{k=<N>} \tilde{x}[n] e^{-jk(\frac{2\pi}{N})n} \qquad (5.2)$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n]e^{-jk(\frac{2\pi}{N})n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(\frac{2\pi}{N})n}$$

- Define the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$a_k = \frac{1}{N}X(e^{jk\omega_0}), \qquad \text{where } \omega_0 = \frac{2\pi}{N} \qquad (5.5)$$

- Combining (5.1) and (5.5) yields $\quad \omega_0 = \frac{2\pi}{N} \rightarrow \frac{1}{N} = \frac{\omega_0}{2\pi}$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0})e^{jk\omega_0 n} \qquad (5.6)$$

$$= \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})\, e^{jk\omega_0 n}\omega_0 \qquad (5.7)$$

- As $N\rightarrow\infty$, $\tilde{x}[n] = x[n]$, $\omega_0 = \frac{2\pi}{N} \rightarrow 0$

$$\lim_{N\rightarrow\infty} \tilde{x}[n] = \lim_{N\rightarrow\infty} \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

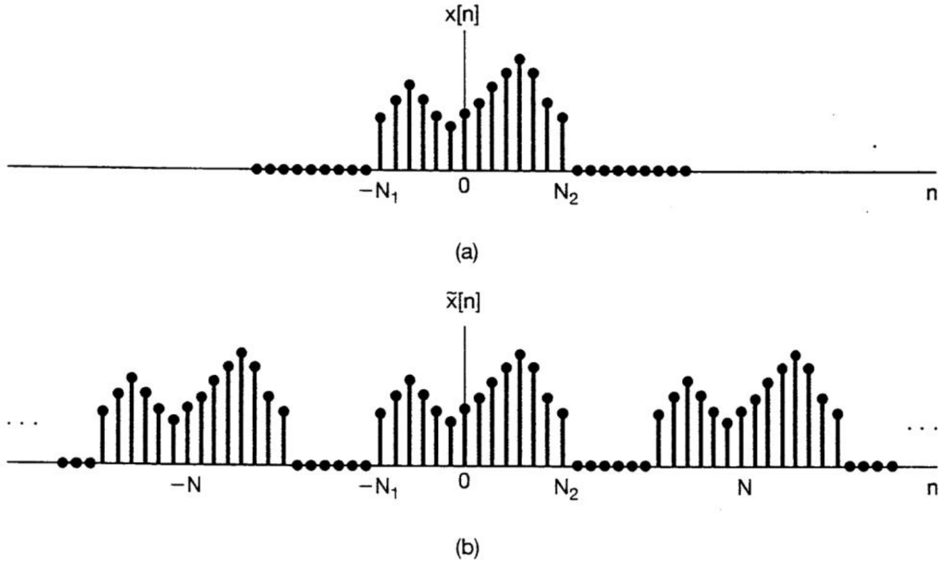


Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

As N increases ω_0 decreases, and as $N \rightarrow \infty$, Eq. (5.7) passes to an integral and $\tilde{x}[n] = x[n]$

- $K = < N >, \omega_0 = \frac{2\pi}{N}$

$$\lim_{N \rightarrow \infty} \tilde{x}[n] = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- $X(e^{j\omega})$, $e^{j\omega n}$, and thus $X(e^{j\omega})e^{j\omega n}$ are periodic with period 2π (Fig 5.2). The interval of integral can be any one with length 2π.

- Discrete-Time Fourier Transform pair

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega \qquad (5.8)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad (5.9)$$

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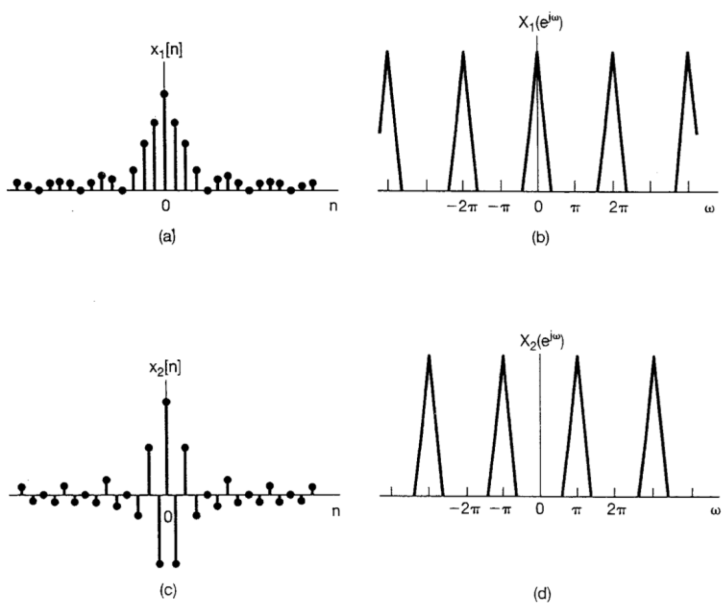

Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$

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Example 5.1 (1/3)

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n]\, e^{-j\omega n}$$

$$\sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure 5.4(a) for $a > 0$ and in figure 5.4(b) for $a < 0$. Note that all of these functions are periodic in ω with period 2π.

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Example 5.1 (2/3)

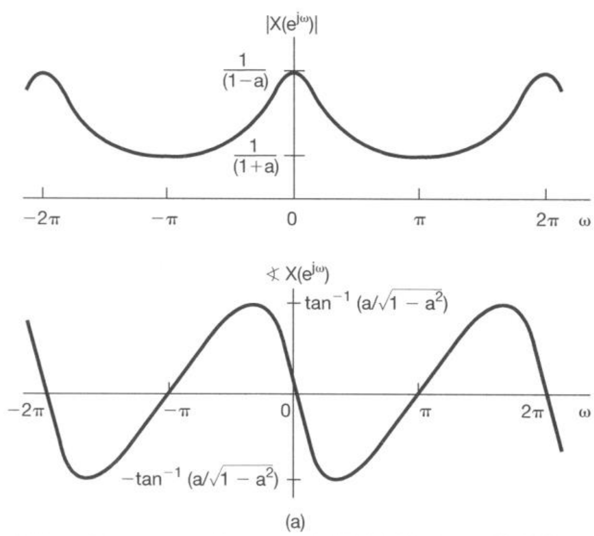

Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

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Example 5.1 (3/3)

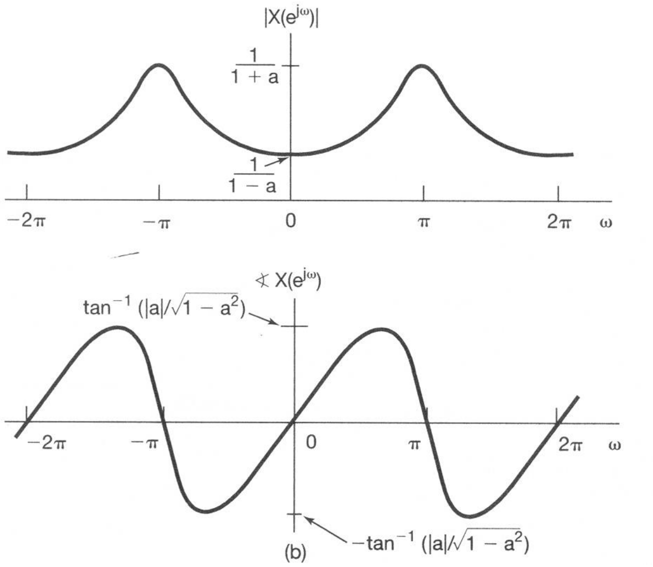

Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

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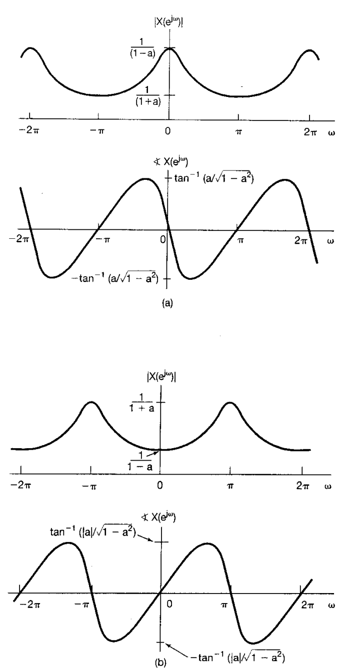

Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

Example 5.2 (1/3)

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier transform is obtained from eq. (5.9):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

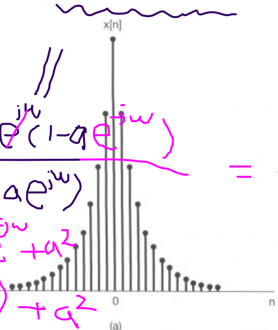

Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform ($0 < a < 1$).

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Example 5.2 (2/3)

- Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m$$

- Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - ae^{j\omega} + ae^{j\omega}(1 - ae^{-j\omega})}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 - a^2}{1 - 2a\,\cos\omega + a^2}$$

- In this case, $X(e^{j\omega})$ is real and is illustrated in Figure 5.5(b), again for $0<a<1$.

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Example 5.2 (3/3)

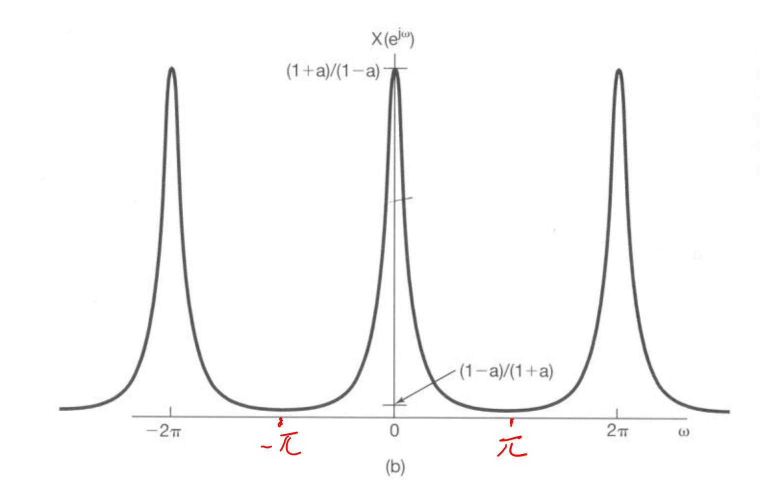

Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform ($0 < a < 1$).

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Example 5.3 (1/2)

- Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \qquad (5.10)$$

which is illustrated in Figure 5.6(a) for $N_1 = 2$.

- In this case,

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}. \qquad (5.11)$$

Let $m = n + N_1$.

$$= \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$$

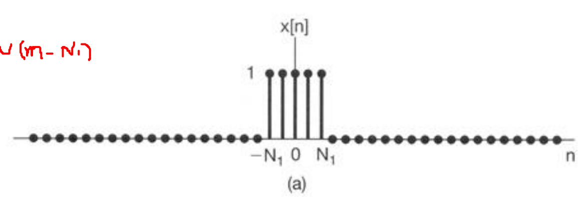

Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Example 5.3 (2/2)

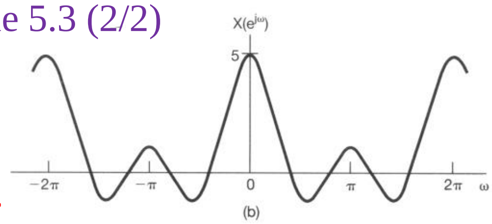

Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Using calculations similar to those employed in obtaining Eq. (3.104) in Example 3.12, p.218 We can write

$$X(e^{j\omega}) = \frac{\sin \omega(N_1 + \frac{1}{2})}{\sin(\omega/2)}. \qquad (5.12)$$

This Fourier transform is sketched in Figure 5.6(b) for N_1=2. The function in eq. (5.12) is the discrete-time counterpart of the sinc function (see Example 4.4). An important difference between these two functions is that the function in eq. (5.12) is periodic with period 2π, whereas the sinc function is aperiodic.

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5.1.3 Convergence Issues Associated with the DTFT

- For an extremely broad class of signals with infinite duration (such as the signals in Example 5.1).
- In this case, again must consider the question of convergence of the infinite summation in the analysis equation (5.9).
- The conditions guarantee the convergence of this sum are direct counterparts of CTFT convergence conditions
- Specifically, eq. (5.9) will converge either if $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \qquad (5.13)$$

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty, \qquad (5.14)$$

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5.1.3 Convergence Issues Associated with the DTFT

- In contrast to the situation for the analysis equation (5.9), there are generally ***no convergence issues*** associated with the synthesis equation (5.8), since the integral in this equation ***is over a finite interval of integration.***

- In particular, if we approximate an aperiodic signal $x[n]$ by an integral of complex exponentials with frequencies taken over the interval $|\omega| \leq W$, i.e.,

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}d\omega, \tag{5.15}$$

then $\hat{x}[n] = x[n]$ for $W = \pi$.

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As in Figure 3.18, we would expect ***not to see*** any behavior like the ***Gibbs phenomenon*** in evaluating the discrete-time Fourier transform synthesis equation.

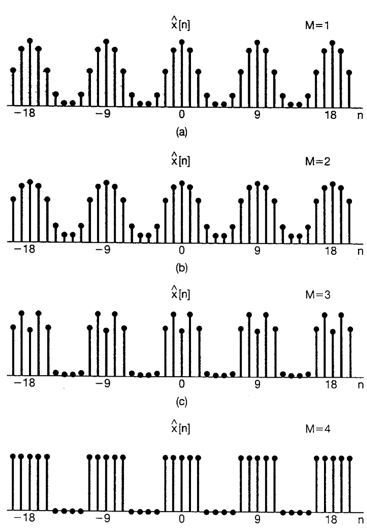

Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

Example 5.4

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum \delta[n]\, e^{-j\omega n} = 1$$

Let $x[n]$ be the unit impulse; that is

$$x[n] = \delta[n]$$

In this case the analysis equation(5.9) is easily evaluated, yielding

$$X(e^{j\omega}) = 1.$$

In other word, just as in continuous time, the unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq.(5.15) to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}. \qquad (5.16)$$

This is plotted in Figure 5.7 for several values of W. As can be seen, the frequency of the oscillations in the approximation increases as W is increased, which is similar to what we observed in the continuous-time case. On the other hand, in contrast to the continuous-time case, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$ as W is increased, and the oscillations disappear entirely for $W = \pi$.

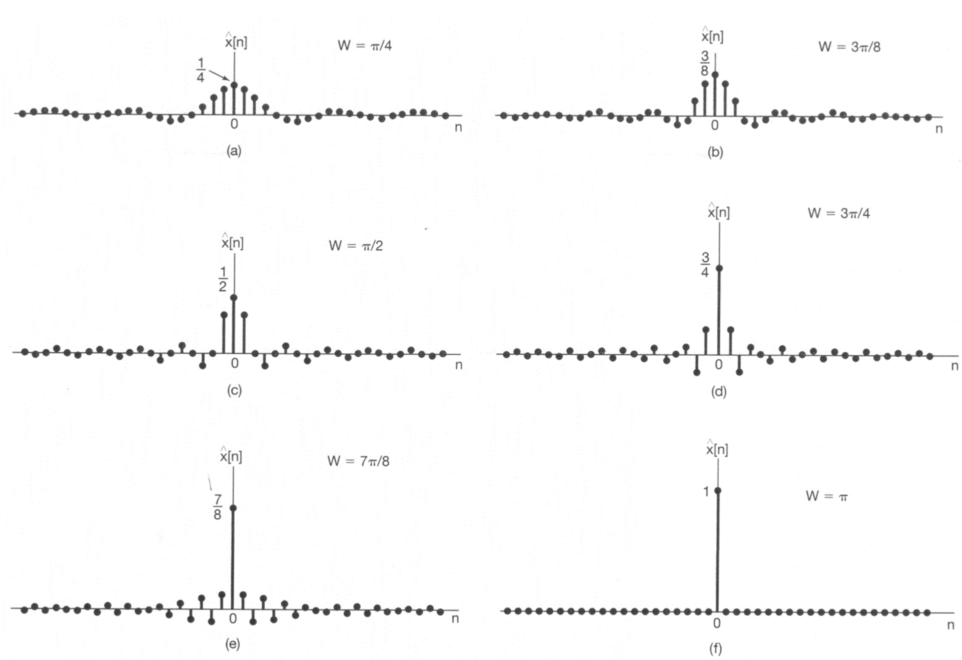

Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.