5.2 The Fourier Transform for Periodic Signals

As in the continuous-time cases, periodic signals can be incorporated within the framework of the discrete-time Fourier transform by ***interpreting the transform of a periodic signal as an impulse train in the frequency domain***. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n} \tag{5.17}$$

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of Eq. (5.17). However, the discrete-time Fourier transform must be *periodic in ω with period 2π*. This then suggests that the Fourier transform of $x[n]$ in Eq. (5.17) should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l), \tag{5.18}$$

which is illustrated in Figure 5.8.

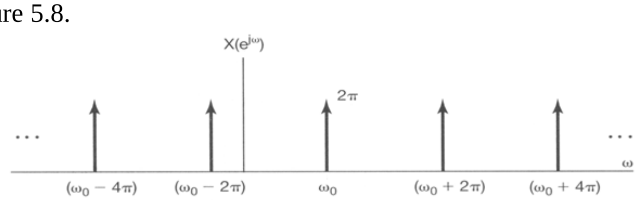

Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

In order to check the validity of this expression, we must evaluate its inverse transform. Substituting Eq. (5.18) into the synthesis Eq. (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)\, e^{j\omega n}d\omega$$

Note that any interval of length 2π includes exactly one impulse in the summation given in Eq. (5.18). Therefore, if the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{5.19}$$

In this case, the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N}) \tag{5.20}$$

so that the Fourier transform of a periodic signal can be ***directly constructed from its Fourier coefficients.***

For continuous-time periodic signals,

$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

- To verify that Eq. (5.20) is in fact correct, note that $x[n]$ in Eq. (5.19) is a linear combination of signals of the form in Eq. (5.17), and thus the Fourier transform of $x[n]$ must be a linear combination of transforms of the form of Eq. (5.18).

- In particular, suppose that we choose the interval of summation in Eq. (5.19) as k=0,1,…, N-1, so that

$$x[n] = a_0 + a_1 e^{j\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1}e^{j(N-1)\left(\frac{2\pi}{N}\right)n}. \tag{5.21}$$

- Thus, $x[n]$ is a linear combination of signals, as in Eq. (5.17), with ω_0=0, $2\pi/N$, $4\pi/N$, …, $(N-1)\,2\pi/N$. The resulting Fourier transform is illustrated in Figure 5.9.

- In Figure 5.9(a), we have depicted the Fourier transform of the first term on the right-hand side of Eq. (5.21):
  - The Fourier transform of the constant signal $a_0 = a_0 e^{j0 \cdot n}$ is a periodic impulse train, as in Eq. (5.18), with ω_0=0 and a scaling of $2\pi a_0$ on each of the impulses.

- From Chapter 4 we know that Fourier series coefficients a_k are periodic with period N, so that $2\pi a_0$= $2\pi a_N$= $2\pi a_{-N}$.

- In Figure 5.9(b) we have illustrated the Fourier transform of the second term in Eq. (5.21), where we have again used Eq. (5.18), in this case for $a_1 e^{j\left(\frac{2\pi}{N}\right)n}$, and the fact that $2\pi a_1$= $2\pi a_{N+1}$= $2\pi a_{-N+1}$.

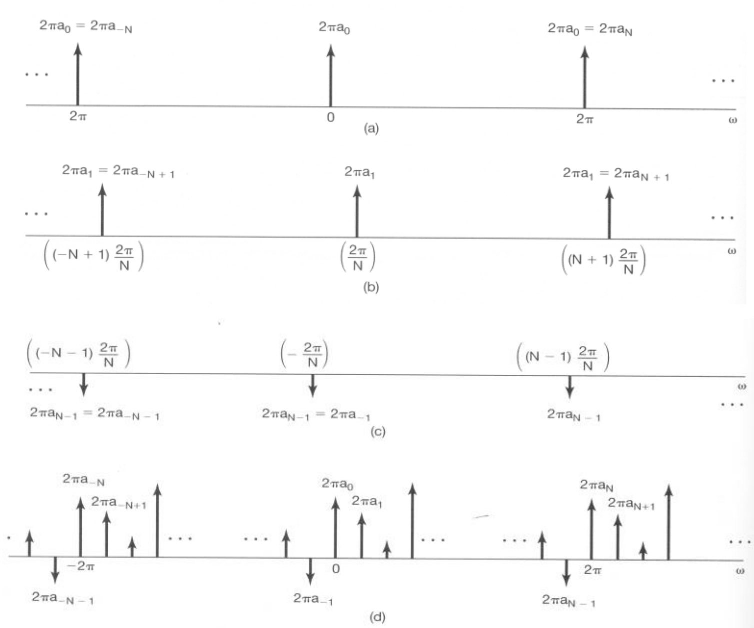

Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

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- Similarly, Fig. 5.9(c) depicts the final term.
- Finally, Fig. 5.9(d) depicts the entire expression for $X\left(e^{j\omega}\right)$

- Because of the ***periodicity*** of the a_k, $X\left(e^{j\omega}\right)$ can be interpreted as a train of impulses occurring at multiples of the fundamental frequency $\omega = 2\pi k/N$, with the area of the impulse located at being $2\pi a_k$.

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right) \qquad (5.20)$$

5.2 The Fourier Transform for Periodic Signals

- Example 5.5

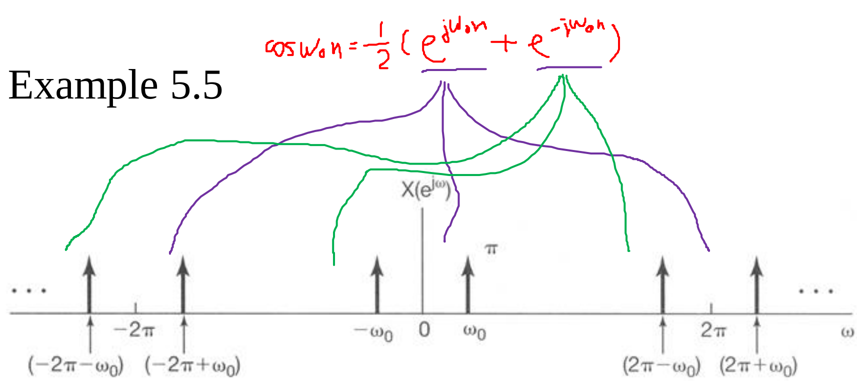

Figure 5.10 Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$, $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$

$= \frac{1}{N} x[n] \cdot 1 = \frac{1}{N}$

- Example 5.6

$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})$

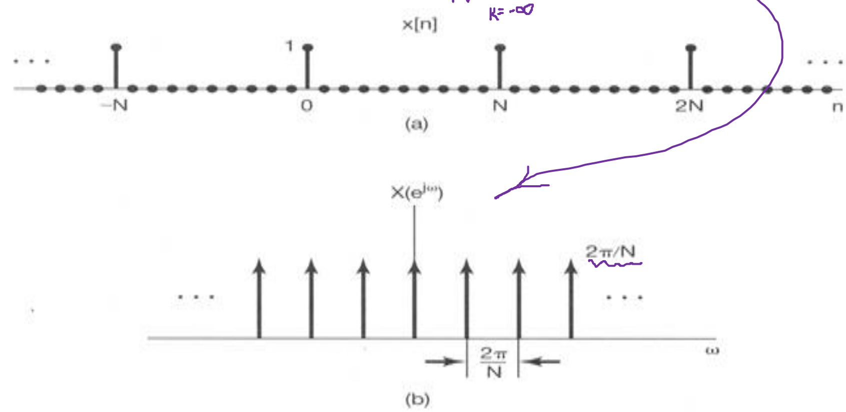

Figure 5.11 (a) Discrete-time periodic impulse train; (b) its Fourier transform.

5.3 Properties of the Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

$$x[n] \xleftrightarrow{\mathcal{FT}} X(e^{j\omega})$$

1. PERIODICITY

As we discussed in Section 5.1, the Discrete-Time Fourier transform is always periodic in ω with period 2π; i, e.,

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

This is in contrast to the continuous-time Fourier Transform, which in general is not periodic.

5.3 Properties of the Discrete-Time Fourier Transform

2. LINEARITY

- If $x_1[n] \xleftrightarrow{\mathcal{FT}} X_1(e^{j\omega})$

and $x_2[n] \xleftrightarrow{\mathcal{FT}} X_2(e^{j\omega})$

➔ $ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{FT}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

3. SHIFTING

- If $x[n] \xleftrightarrow{\mathcal{FT}} X(e^{j\omega})$

then $x[n-n_0] \xleftrightarrow{\mathcal{FT}} e^{-j\omega n_0} X(e^{j\omega})$

and $e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{FT}} X(e^{j(\omega-\omega_0)})$

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5.3 Properties of the Discrete-Time Fourier Transform

- Example 5.7

$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$

$h_{hp}[n] = h_{lp}[n]e^{j\pi n}$

$= (-1)^n h_{lp}[n]$

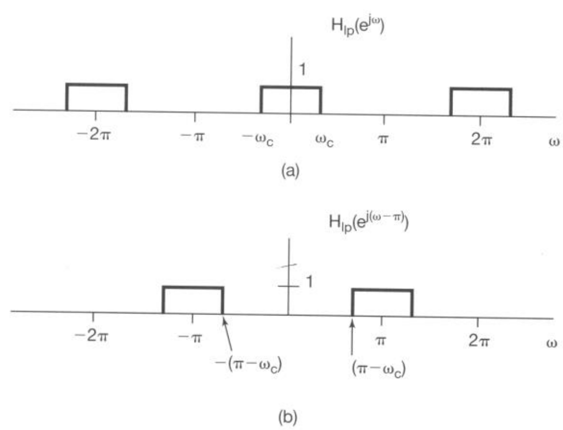

Figure 5.12 (a) Frequency response of a lowpass filter; (b) frequency response of a highpass filter obtained by shifting the frequency response in (a) by $\omega = \pi$ corresponding to one-half period.

5.3 Properties of the Discrete-Time Fourier Transform

- **Conjugation and Conjugate Symmetry (1/2)**

If $x[n] \overset{\mathcal{FT}}{\longleftrightarrow} X(e^{j\omega})$

then $x^*[n] \overset{\mathcal{FT}}{\longleftrightarrow} X^*(e^{-j\omega})$

If $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is $X(e^{j\omega}) = X^*(e^{-j\omega})$

From this, it follows that $\mathrm{Re}\{X(e^{j\omega})\}$ is an even function of ω and $\mathrm{Im}\{X(e^{j\omega})\}$ an odd function of ω.

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5.3 Properties of the Discrete-Time Fourier Transform

- **Conjugation and Conjugate Symmetry (2/2)**

$$\mathcal{E}v\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \Re e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\Im m\{X(e^{j\omega})\}$$

where $\mathcal{E}v$ and $\mathcal{O}d$ denote the even and odd parts, respectively.

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5.3 Properties of the Discrete-Time Fourier Transform

- **Differencing and Accumulation**

1. First difference :

$$x[n]-x[n-1] \xleftrightarrow{\mathcal{FT}} (1-e^{-j\omega})X(e^{j\omega})$$

2. Summation:

ω=0, dc value

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{\mathcal{FT}} \frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

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5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.8

Let us derive the Fourier transform $X(e^{j\omega})$ of the unit step $x[n] = u[n]$ by making use of the accumulation property and the knowledge that

$$g[n] = \delta[n] \xleftrightarrow{\mathfrak{F}} G(e^{j\omega}) = 1.$$

From Section 1.4.1 we know that the unit step is the running sum of the unit impulse. That is,

$$x[n] = \sum_{m=-\infty}^{n} g[m].$$

Taking the Fourier transform of both sides and using accumulation yields

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})}G(e^{j\omega}) + \pi G(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$= \frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k).$$

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5.3 Properties of the Discrete-Time Fourier Transform

- **Time Reversal**

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n]=x[-n]$ From eq.(5.9),

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}$$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Substituting $m=-n$ into eq.(5.40),we obtain

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega})$$

That is ,

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

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5.3 Properties of the Discrete-Time Fourier Transform

In C-T FT, we have time scaling property.

- **Time Expansion**
- Recall in continuous-time case , where *a* is a real number.

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- In constructing similar $x[an]$, note that:

$a = 0.3$, $n = 1$

$x[an] = x[0.3]$

(i) *a* has to be an integer; otherwise, $x[an]$ is not defined in general.

(ii) If a is an integer, say 2, then in $x[2n]$ all odd sample will be lost.

- Instead, we define a “slow down” version of $x[n]$.

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], \text{ if } n \text{ is a multiple of } k \\ 0, \quad \text{otherwise} \end{cases}$$

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5.3 Properties of the Discrete-Time Fourier Transform

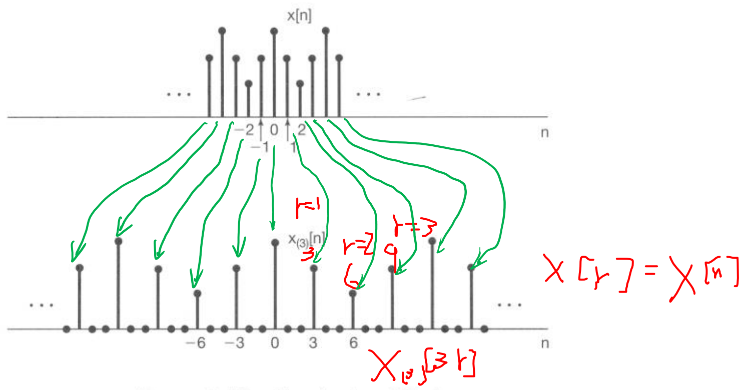

Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

k=3

5.3 Properties of the Discrete-Time Fourier Transform

- **Time Expansion**

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$x_{(k)}[n] \overset{FT}{\leftrightarrow} X(e^{jk\omega})$$

5.3 Properties of the Discrete-Time Fourier Transform

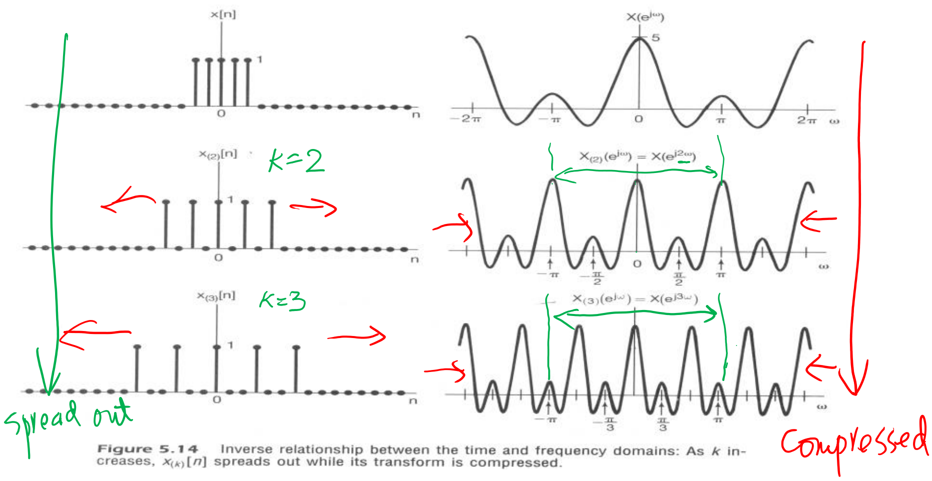

Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.9 (1/3)

As an illustration of the usefulness of the time-expansion property in determining Fourier transforms, let us consider the sequence $x[n]$ displayed in Figure 5.15(a). This sequence can be related to the simpler sequence $y[n]$ depicted in Figure 5.15(b). In particular

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

where

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

and $y_{(2)}[n-1]$ represents $y_{(2)}[n]$ shifted one unit to the right. The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in Figures 5.15(c) and (d), respectively.

Next, note that $y[n] = g[n-2]$, where $g[n]$ is a rectangular pulse as considered in Example 5.3 (with $N_1 = 2$) and as depicted in Figure 5.6(a). Consequently, from Example 5.3 and the time-shifting property, we see that

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.9 (2/2)

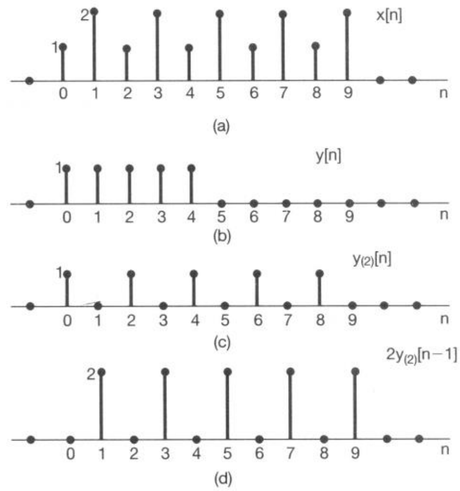

Figure 5.15 (a) The signal $x[n]$ in Example 5.9; (b) the signal $y[n]$; (c) the signal $y_{(2)}[n]$ obtained by inserting one zero between successive values of $y[n]$; and (d) the signal $2y_{(2)}[n-1]$.

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5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.9 (3/3)

Using the time-expansion property, we then obtain

$$y_{(2)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)},$$

and using the linearity and time-shifting properties, we get

$$2y_{(2)}[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}.$$

Combining these two results, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

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5.3 Properties of the Discrete-Time Fourier Transform

- **Differentiation in Frequency**

Again, let

$$x[n] \overset{FT}{\leftrightarrow} X(e^{j\omega})$$

If we use the definition of $X(e^{j\omega})$ in the analysis Eq. (5.9) and differentiate both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of this equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$nx[n] \overset{FT}{\leftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}. \qquad (5.46)$$

5.3 Properties of the Discrete-Time Fourier Transform

- **Parseval's Relation**

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

1. The left-hand side of this equation is the total energy in the signal $x[n]$.

2. $|X(e^{j\omega})|^2$ is referred to as the energy-time-density spectrum of the signal $x[n]$.

This equation is the counterpart for aperiodic signal of Parseval's relation.

5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.10 (1/3)

Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \leq \omega \leq \pi$ in Figure 5.16. We wish to determine whether or not, in the time domain, $x[n]$ is periodic, real, even, and/or of finite energy.

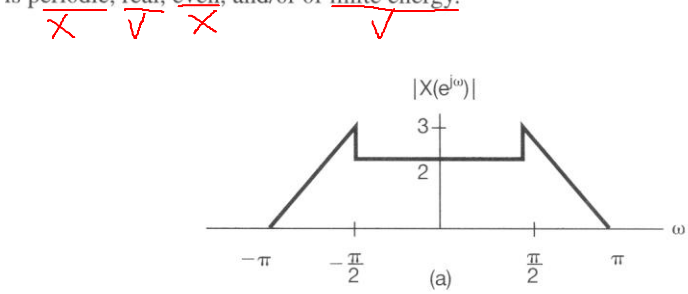

Figure 5.16 Magnitude and phase of the Fourier transform for Example 5.10.

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega = \frac{7}{4}$$

5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.10 (2/3)

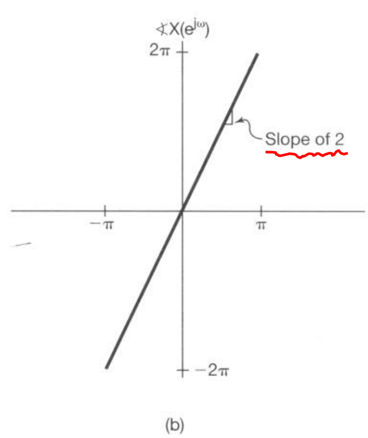

Figure 5.16 Magnitude and phase of the Fourier transform for Example 5.10.

5.3 Properties of the Discrete-Time Fourier Transform

EXAMPLE 5.10 (3/3)

Accordingly, we note first that periodicity in the time domain implies that the Fourier transform is zero, except possibly for impulses located at various integer multiples of the fundamental frequency. This is not true for $X(e^{j\omega})$. We conclude, then, that $x[n]$ is *not* periodic.

Next, from the symmetry properties for Fourier transforms, we know that a real-valued sequence must have a Fourier transform of even magnitude and a phase function that is odd. This is true for the given $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$. We thus conclude that $x[n]$ is real.

Third, if $x[n]$ is an even function, then, by the symmetry properties for real signals, $X(e^{j\omega})$ must be real and even. However, since $X(e^{j\omega}) = |X(e^{j\omega})|e^{-j2\omega}$, $X(e^{j\omega})$ is not a real-valued function. Consequently, $x[n]$ is not even.

Finally, to test for the finite-energy property, we may use Parseval's relation,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.$$

It is clear from Figure 5.16 that integrating $|X(e^{j\omega})|^2$ from $-\pi$ to π will yield a finite quantity. We conclude that $x[n]$ has finite energy.

5.4 The Convolution Property

- **The Convolution Property**

$$y[n] = x[n] * h[n]$$

and

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

5.4 The Convolution Property

EXAMPLE 5.11

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0].$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}.$$

Thus, for any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega}). \tag{5.49}$$

We note that, for this example, $y[n] = x[n - n_0]$ and eq. (5.49) is consistent with the time-shifting property. Note also that the frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ of a pure time shift has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency.

5.4 The Convolution Property

EXAMPLE 5.12 (1/2)

Consider the discrete-time ideal lowpass filter introduced in Section 3.9.2. This system has the frequency response $H(e^{j\omega})$ illustrated in Figure 5.17(a). Since the impulse response and frequency response of an LTI system are a Fourier transform pair, we can determine the impulse response of the ideal lowpass filter from the frequency response using the Fourier transform synthesis equation (5.8). In particular, using $-\pi \le \omega \le \pi$ as the interval of integration in that equation, we see from Figure 5.17(a) that

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{\sin \omega_c n}{\pi n}, \end{aligned} \tag{5.50}$$

which is shown in Figure 5.17(b).

5.4 The Convolution Property

EXAMPLE 5.12 (2/2)

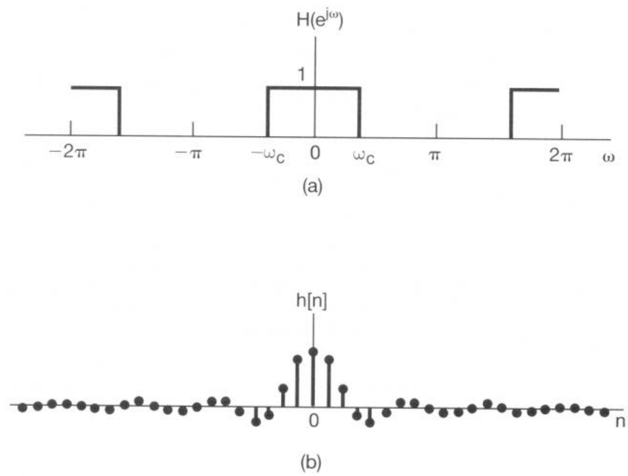

Figure 5.17 (a) Frequency response of a discrete-time ideal lowpass filter; (b) impulse response of the ideal lowpass filter.

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5.4 The Convolution Property

EXAMPLE 5.13 (1/4)

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n],$$

with $|\alpha| < 1$, and suppose that the input to this system is

$$x[n] = \beta^n u[n],$$

with $|\beta| < 1$. Evaluating the Fourier transforms of $h[n]$ and $x[n]$, we have

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \tag{5.51}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}, \tag{5.52}$$

so that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}. \tag{5.53}$$

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5.4 The Convolution Property

EXAMPLE 5.13 (2/4)

As with Example 4.19, determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of partial fractions. Specifically, $Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$, and we would like to express this as a sum of simpler terms of this type so that we can find the inverse transform of each term by inspection (together, perhaps, with the use of the frequency differentiation property of Section 5.3.8). The general algebraic procedure for rational transforms is described in the appendix. For this example, if $\alpha \neq \beta$, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}. \tag{5.54}$$

Equating the right-hand sides of eqs (5.53) and (5.54), we find that

$$A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}.$$

Therefore, from Example 5.1 and the linearity property, we can obtain the inverse transform of eq. (5.54) by inspection:

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \\ &= \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]. \end{aligned} \tag{5.55}$$

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5.4 The Convolution Property

EXAMPLE 5.13 (3/4)

For $\alpha = \beta$, the partial-fraction expansion in eq. (5.54) is not valid. However, in this case,

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2,$$

which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha}e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right). \tag{5.56}$$

As in Example 4.19, we can use the frequency differentiation property, eq. (5.46), together with the Fourier transform pair

$$\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}},$$

to conclude that

$$n\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

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5.4 The Convolution Property

EXAMPLE 5.13 (4/4)

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1}u[n+1] \overset{\mathfrak{F}}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right),$$

and finally, accounting for the factor $1/\alpha$, in eq. (5.56), we obtain

$$y[n] = (n+1)\alpha^n u[n+1]. \tag{5.57}$$

It is worth noting that, although the right-hand side is multiplied by a step that begins at $n = -1$, the sequence $(n+1)\alpha^n u[n+1]$ is still zero prior to $n = 0$, since the factor $n+1$ is zero at $n = -1$. Thus, we can alternatively express $y[n]$ as

$$y[n] = (n+1)\alpha^n u[n]. \tag{5.58}$$

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5.4 The Convolution Property

EXAMPLE 5.14 (1/3)

Consider the system shown in Figure 5.18(a) with input $x[n]$ and output $y[n]$. The LTI systems with frequency response $H_{lp}(e^{j\omega})$ are ideal lowpass filters with cutoff frequency $\pi/4$ and unity gain in the passband.

Let us first consider the top path in Figure 5.18(a). The Fourier transform of the signal $w_1[n]$ can be obtained by noting that $(-1)^n = e^{j\pi n}$ so that $w_1[n] = e^{j\pi n}x[n]$. Using the frequency-shifting property, we then obtain

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)}).$$

The convolution property yields

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

Since $w_3[n] = e^{j\pi n}w_2[n]$, we can again apply the frequency-shifting property to obtain

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}). \end{aligned}$$

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5.4 The Convolution Property

EXAMPLE 5.14 (2/3)

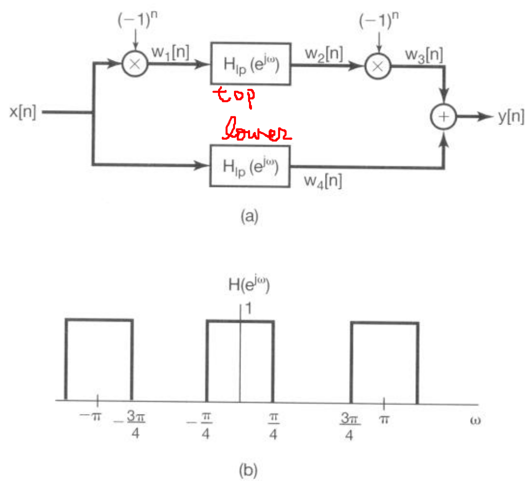

Figure 5.18 (a) System interconnection for Example 5.14; (b) the overall frequency response for this system.

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5.4 The Convolution Property

EXAMPLE 5.14 (3/3)

Since discrete-time Fourier transforms are always periodic with period 2π,

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

Applying the convolution property to the lower path, we get

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega}).$$

From the linearity property of the Fourier transform, we obtain

$$\begin{aligned} Y(e^{j\omega}) &= W_3(e^{j\omega}) + W_4(e^{j\omega}) \\ &= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}). \end{aligned}$$

Consequently, the overall system in Figure 5.18(a) has the frequency response

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

which is shown in Figure 5.18(b).

As we saw in Example 5.7, $H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of an ideal highpass filter. Thus, the overall system passes both low and high frequencies and stops frequencies between these two passbands. That is, the filter has what is often referred to as an *ideal bandstop characteristic*, where the stopband is the region $\pi/4 < |\omega| < 3\pi/4$.

5.5 The Multiplication Property

- Let $y[n] = x_1[n]x_2[n]$

Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]\, e^{-j\omega n}$$

So

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n]\, \{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\}e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\{\int_{2\pi} X_1(e^{j\theta})[\sum_{n=-\infty}^{\infty} x_{2[n]}\, e^{-j(\omega-\theta)n}]d\theta\}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

5.5 The Multiplication Property

EXAMPLE 5.15 (1/3)

Consider the problem of finding the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ which is the product of two other signals; that is,

$$x[n] = x_1[n]x_2[n],$$

where

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

and

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

From the multiplication property given in eq. (5.63), we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, where the integral in eq. (5.63) can be taken over any interval of length 2π. Choosing the interval $-\pi < \theta \leq \pi$, we obtain

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \quad (5.64)$$

5.5 The Multiplication Property

EXAMPLE 5.15 (2/3)

Equation (5.64) resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \leq \pi$. However, we can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}.$$

Then, replacing $X_1(e^{j\theta})$ in eq. (5.64) by $\widehat{X}_1(e^{j\theta})$, and using the fact that $\widehat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we see that

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \end{aligned}$$

Thus, $X(e^{j\omega})$ is $1/2\pi$ times the aperiodic convolution of the rectangular pulse $\hat{X}_1(e^{j\omega})$ and the periodic square wave $X_2(e^{j\omega})$, both of which are shown in Figure 5.19. The result of this convolution is the Fourier transform $X(e^{j\omega})$ shown in Figure 5.20.

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5.5 The Multiplication Property

EXAMPLE 5.15 (3/3)

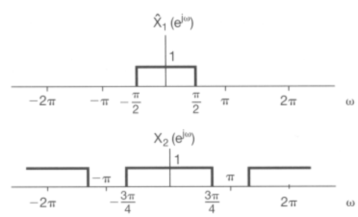

Figure 5.19 $\hat{X}_1(e^{j\omega})$ representing one period of $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$. The linear convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$ corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.

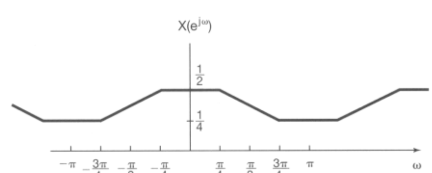

Figure 5.20 Result of the periodic convolution in Example 5.15.

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5.6 Tables of Fourier Transform Properties and Basic Fourier Transform Pairs

- In Table 5.1, we summarize a number of important properties of the discrete-time Fourier transform and indicate the section of the text in which each is discussed.
- In Table 5.2, we summarize some of the basic and most important discrete-time Fourier transform pairs. Many of these have been derived in examples in the chapter.

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TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$

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(followed by the next page)

5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

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TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$

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(followed by the next page)

Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1+1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

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5.7 Duality

- **Duality in Discrete-time Fourier series**

Consider two periodic sequences with period *N*, related through the summation

$a_k \longleftrightarrow x[n]$ Refer to (3.95)

$$f[m] = \frac{1}{N} \sum_{r=<N>} g[r] e^{-jr\left(\frac{2\pi}{N}\right)m}$$

- **Let *m*=*k*, *r*=*n*.**

$$f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \quad (5.65)$$

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

$$g[n] \overset{FS}{\longleftrightarrow} f[k] \quad (5.66)$$

- If we **let *m*=*n* and *r*=-*k***, Eq. (5.65) becomes

Eq (3.94)

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k] e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$f[n] \overset{FS}{\longleftrightarrow} \frac{1}{N} g[-k] \quad (5.67)$$

$x[n]$ a_k

5.7 Duality

$$x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0} \quad (5.68)$$

and $$e^{jm\left(\frac{2\pi}{N}\right)n} x[n] \xleftrightarrow{FS} a_{k-m} \quad (5.69)$$

$$\sum_{r=<N>} x[r]y[n-r] \xleftrightarrow{FS} N a_k b_k \quad (5.70)$$

$$x[n]y[n] \xleftrightarrow{FS} \sum_{l=<N>} a_l b_{k-l} \quad (5.71)$$

5.7 Duality

EXAMPLE 5.16 (1/2)

Consider the following periodic signal with a period of $N = 9$:

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \frac{5}{9}, & n = \text{multiple of } 9 \end{cases} \quad (5.72)$$

In Chapter 3, we found that a rectangular square wave has Fourier coefficients in a form much as in eq. (5.72). Duality, then, suggests that the coefficients for $x[n]$ must be in the form of a rectangular square wave. To see this more precisely, let $g[n]$ be a rectangular square wave with period $N = 9$ such that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4. \end{cases}$$

The Fourier series coefficients b_k for $g[n]$ can be determined from Example 3.12 as

$$b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \frac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

The Fourier series analysis equation (3.95) for $g[n]$ can now be written as

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}.$$

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5.7 Duality

EXAMPLE 5.16 (2/2)

Interchanging the names of the variables k and n and noting that $x[n] = b_n$, we find that

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Letting $k' = -k$ in the sum on the right side, we obtain

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2} e^{+j2\pi nk'/9}.$$

Finally, moving the factor 1/9 inside the summation, we see that the right side of this equation has the form of the synthesis equation (3.94) for $x[n]$. We thus conclude that the Fourier coefficients of $x[n]$ are given by

$$a_k = \begin{cases} 1/9, & |k| \le 2 \\ 0, & 2 < |k| \le 4, \end{cases}$$

and, of course, are periodic with period $N = 9$.

5.7 Duality

Duality between the Discrete-Time Fourier Transform and the Continuous-Time Fourier Series

[eq. (5.8)]

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad (5.73)$$

[eq. (5.9)]

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (5.74)$$

[eq. (5.38)]

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad (5.75)$$

[eq. (5.39)]

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad (5.76)$$

5.7 Duality

EXAMPLE 5.17 (1/2)

The duality between the discrete-time Fourier transform synthesis equation and the continuous-time Fourier series analysis equation may be exploited to determine the discrete-time Fourier transform of the sequence

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

To use duality, we first must identify a continuous-time signal $g(t)$ with period $T = 2\pi$ and Fourier coefficients $a_k = x[k]$. From Example 3.5, we know that if $g(t)$ is a periodic square wave with period 2π (or, equivalently, with fundamental frequency $\omega_0 = 1$) and with

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases},$$

then the Fourier series coefficients of $g(t)$ are

5.7 Duality

EXAMPLE 5.17 (2/2)

$$a_k = \frac{\sin(kT_1)}{k\pi}$$

Consequently, if we take $T_1 = \pi/2$, we will have $a_k = x[k]$. In this case the analysis equation for $g(t)$ is

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jkt}dt.$$

Renaming k as n and t as ω, we have

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jn\omega}d\omega. \tag{5.77}$$

Replacing n by $-n$ on both sides of eq. (5.77) and noting that the sinc function is even, we obtain

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{jn\omega}d\omega.$$

The right-hand side of this equation has the form of the Fourier transform synthesis equation for $x[n]$, where

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}.$$

5.7 Duality

TABLE 5.3

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality (Fourier Transform: continuous time ↔ continuous frequency); duality (Fourier Series frequency domain ↔ discrete-time Fourier transform time domain); duality (discrete time Fourier series: time domain ↔ frequency domain)

5.8 Systems Characterized By Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{5.78}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \tag{5.79}$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \tag{5.80}$$

EXAMPLE 5.18

Consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n], \tag{5.81}$$

time shift

with $|a| < 1$. From eq. (5.80), the frequency response of this system is

F.T.

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{5.82}$$

Comparing this with Example 5.1, we recognize it as the Fourier transform of the sequence $a^n u[n]$. Thus, the impulse response of the system is

$$h[n] = a^n u[n]. \tag{5.83}$$

$$Y(e^{j\omega}) - ae^{-j\omega}\,Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega})(1 - ae^{-j\omega}) = X(e^{j\omega}) \quad H = \frac{Y}{X}$$

EXAMPLE 5.19

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]. \tag{5.84}$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}. \tag{5.85}$$

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \tag{5.86}$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \tag{5.87}$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \tag{5.88}$$

EXAMPLE 5.20 (1/2)

Consider the LTI system of Example 5.19, and let the input to this system be

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Then, using eq. (5.80) and Example 5.1 or 5.18, we obtain

$$\begin{aligned} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) &= \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right] \\ &= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2}. \end{aligned} \tag{5.89}$$

As described in the appendix, the form of the partial-fraction expansion in this case is

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \tag{5.90}$$

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EXAMPLE 5.20(2/2)

where the constants B_{11}, B_{12}, and B_{21} can be determined using the techniques described in the appendix. This particular expansion is worked out in detail in Example A.4, and the values obtained are

$$B_{11} = -4, \quad B_{12} = -2, \quad B_{21} = 8,$$

so that

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}. \tag{5.91}$$

The first and third terms are of the same type as those encountered in Example 5.19, while the second term is of the same form as one seen in Example 5.13. Either from these examples or from Table 5.2, we can invert each of the terms in eq. (5.91) to obtain the inverse transform

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n]. \tag{5.92}$$

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- 5.32. The Fourier transform of a discrete-time signal $x[n]$ is

  $$X(e^{j\omega}) = e^{j2\omega}\left(1 - e^{-j3\omega}\right).$$

  Completely specify $x[n]$.