Signal and Systems

Chapter 7

Sampling

- Under some conditions, a continuous-time signal can be *completely represented* by and *recoverable* from knowledge of its values, *at points* ***equally spaced*** (not always) *in time*.

- Exploit sampling to convert a continuous-time signal to a ***discrete-time signal***, process the discrete-time signal using a discrete-time system, and then convert back to continuous time.

- Processing discrete-time signals is ***more flexible*** and is ***often preferable*** to processing continuous-time signals.
- This is due to in large part to the *dramatic development of digital technology over the past few decades*
- Resulting in the availability of ***inexpensive, lightweight, programmable, and easily reproducible discrete-time systems***.

7.1 Representation of a continuous-time signal by its samples: The sample theorem

- Figure 7.1, three *different* continuous-time signals have *identical* values at integer multiples of *T.*

$$x_1(kT) = x_2(kT) = x_3(kT)$$
$$\text{But } x_1(t) \neq x_2(t) \neq x_3(t)$$

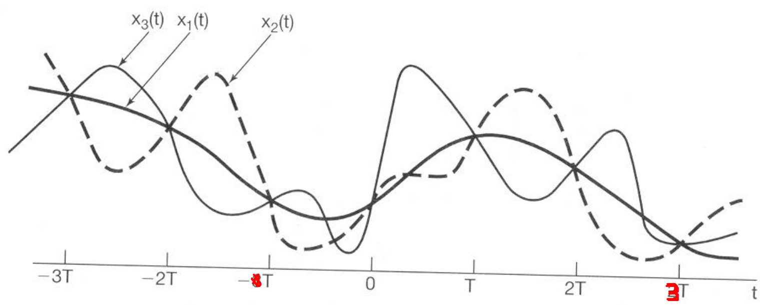


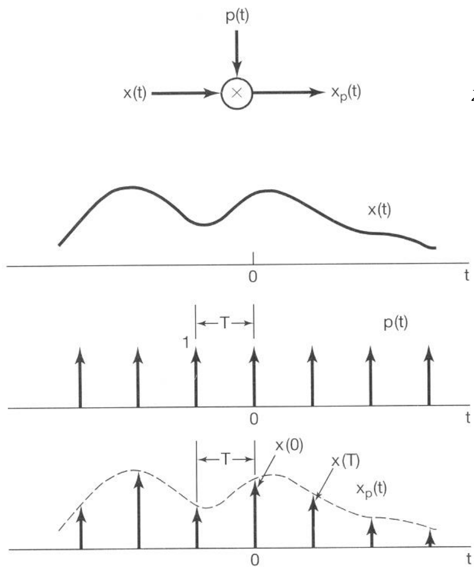
Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

7.1.1 Impulse-Train Sampling

- T: sampling period
- $\omega_s = 2\pi/T$: fundamental frequency
- $p(t)$: sampling function
- $x_p(t)=x(t)p(t)$, where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

7.1.1 Impulse-Train Sampling



$x_p(t)=x(t)p(t)$

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Figure 7.2 Impulse-train sampling.

$$x_p(t)=x(t)p(t)$$

$$=\sum_{n=-\infty}^{\infty}x(t)\delta(t-nT)=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$$

$$X_p(j\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)P(j(\omega-\theta))d\theta$$

From Example 4.8, p. 299, $P(j\omega)$

$$=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))$$

$w_s=\frac{2\pi}{T}$

T↑ w_s↓

T↓ w_s↑

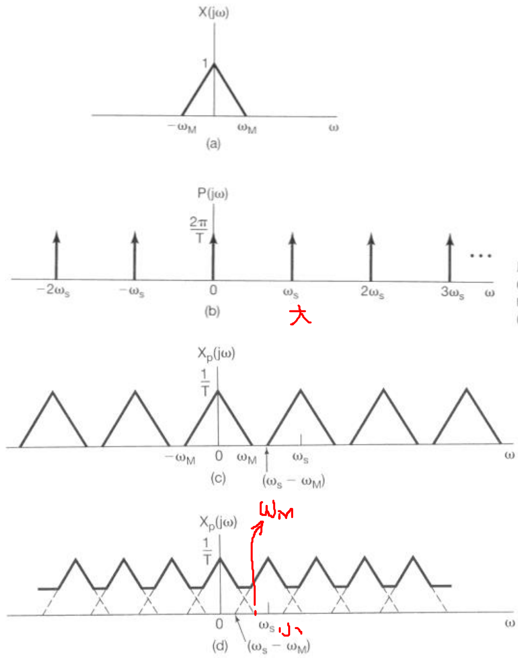


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

Impulse-Train Sampling

- Figure 7.3(c):

$$\omega_M < (\omega_s - \omega_M)$$
$$\omega_s > 2\omega_M$$

- There is ***no overlap*** between the shifted replicas of $X(j\omega)$
- Figure 7.3(d): there is ***overlap*** between the shifted replicas

$$\omega_M > (\omega_s - \omega_M)$$
$$\omega_s < 2\omega_M$$

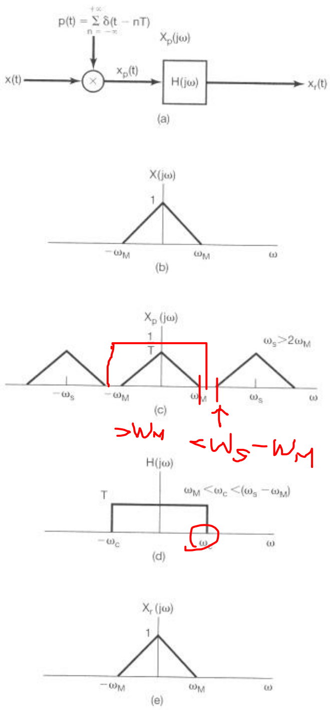


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $x_r(t)$.

If $\omega_s > 2\omega_M$, $x(t)$ can be recovered exactly from $x_p(t)$ by means of a ***lowpass*** filter with gain T and a cutoff frequency ***greater than*** ω_M and ***less than*** $\omega_s - \omega_M$.

Sampling theorem:

- Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \dots,$ if

$$\omega_s > 2\omega_M,$$

where
$$\omega_s = \frac{2\pi}{T}.$$

- Given these samples, we can reconstruct $x(t)$ by *generating a periodic impulse train* in which successive impulses have amplitudes that are successive sample values.

- This impulse train is then processed through an *ideal lowpass filter* with gain T and cutoff frequency ***greater than*** $\boldsymbol{\omega_M}$ ***and less than*** $\boldsymbol{\omega_s - \omega_M}$. The resulting output signal will *exactly equal* $x(t)$.

- The frequency $2\omega_M$, which under the sampling theorem, must be exceeded by the sampling frequency ω_s, is commonly referred to as the ***Nyquist rate***.

7.1.2 Sampling with a Zero-Order Hold

- In practice, narrow, large-amplitude pulses are *relatively difficult to generate and transmit*.
- It is often more convenient to generate the sampled signal in a form of referred to as a ***zero-order hold***.
- Such a system samples $x(t)$ at a given instant and holds that value until the next instant at which a sample is taken (Fig. 7.5)

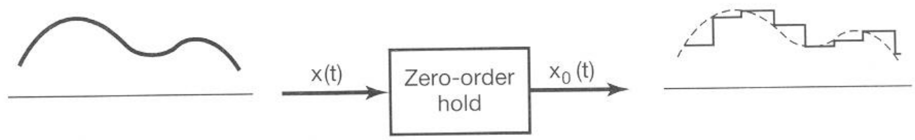


Figure 7.5 Sampling utilizing a zero-order hold.

- The output $x_0(t)$ of the zero-order hold can be generated by impulse-train sampling *followed by an LTI system with a rectangular impulse response* (Fig. 7.6)

Sampling with a Zero-Order Hold

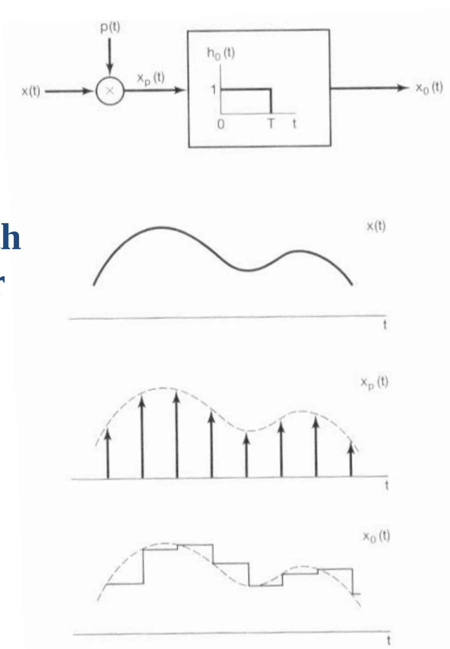


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.

- To reconstruct $x(t)$ from $x_0(t)$, we consider processing $x_0(t)$ with an LTI system with impulse response $h_r(t)$ and frequency response $H_r(j\omega)$. (Fig. 7.7)
- We wish to specify $H_r(j\omega)$ so that $r(t) = x(t)$

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}\right], \qquad (7.7)$$

$$H_r(j\omega) = \frac{e^{\frac{j\omega T}{2}} H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}, \qquad (7.8)$$

$$H_0(j\omega)H_r(j\omega) = H(j\omega)$$

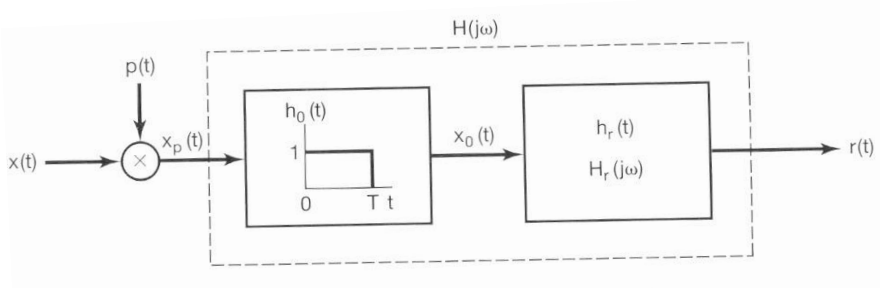


Figure 7.7 Cascade of the representation of a zero-order hold (Figure 7.6) with a reconstruction filter.

If the cutoff frequency of $H(j\omega)$ is $\omega_s/2$, the ideal magnitude and phase for the reconstruction filter ***following a zero-order hold*** is shown below:

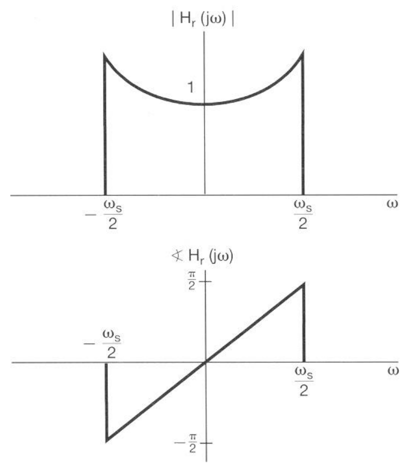


Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

Sampling with a Zero-Order Hold

In practice the frequency response in Eq. (7.8) cannot be exactly realized, and thus an adequate approximation to it must be designed.

In fact, in many situations, the output of the zero-order hold is considered an adequate approximation to the original signal by itself, without any additional lowpass filtering, and present a possible, although admittedly very coarse, interpolation between the sample values.

7.2 Reconstruction of a signal from its samples using interpolation

- ***Interpolation***: the fitting of a continuous signal of a set of sample values
  - Zero-order hold
  - Linear interpolation (Fig. 7.9)

Figure 7.9 Linear interpolation between sample points. The dashed curve represents the original signal and the solid curve the linear interpolation.

Back to Fig. 7.4,

$$x_r(t) = x_p(t) * h(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$$

For ideal lowpass filter $H(j\omega)$ in Fig. 7.4

$$h(t) = \frac{\omega_c T sin(\omega_c t)}{\pi \omega_c t},$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\frac{sin(\omega_c(t-nT))}{\omega_c(t-nT)}.$$

$= x(t)$

Reconstruction of a signal from its samples using interpolation

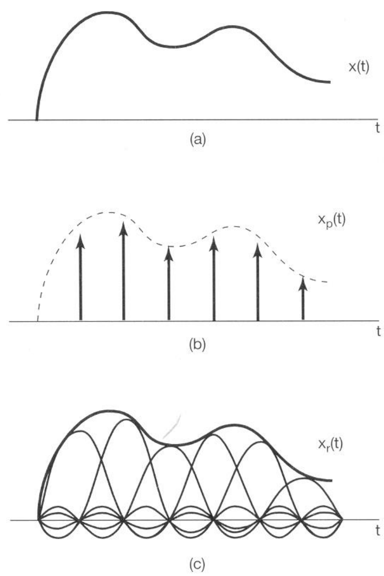


Figure 7.10 Ideal band-limited interpolation using the sinc function: (a) band-limited signal $x(t)$; (b) impulse train of samples of $x(t)$; (c) ideal band-limited interpolation in which the impulse train is replaced by a superposition of sinc functions [eq. (7.11)].

Reconstruction of a signal from its samples using interpolation

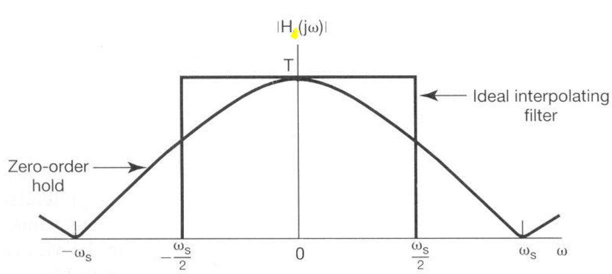


Figure 7.11 Transfer function for the zero-order hold and for the ideal interpolating filter.

- The zero-order hold is a ***very rough*** approximation, although in some cases it is sufficient.
- Higher order holds: a variety of smoother interpolation strategies.

Reconstruction of a signal from its samples using interpolation under different sampling frequencies

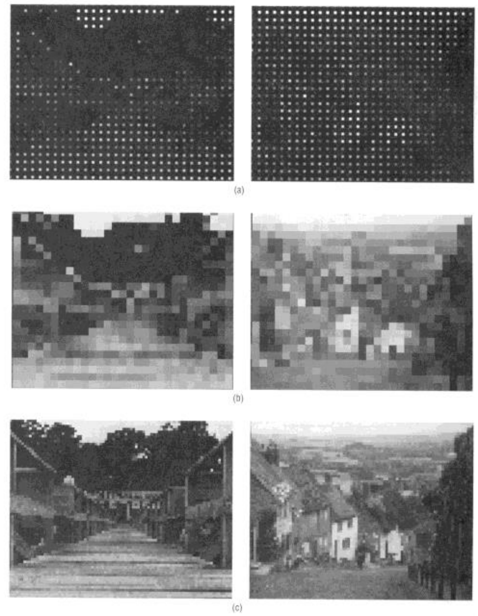
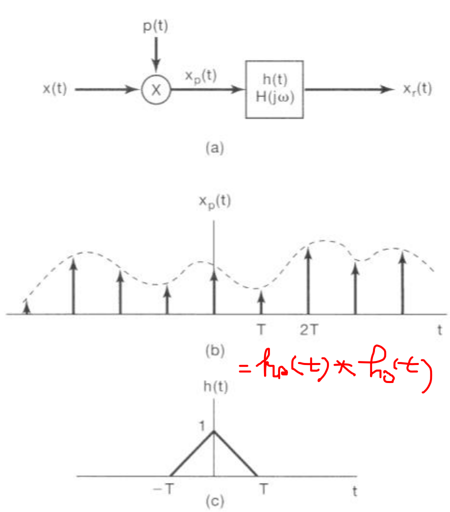


Figure 7.12 (a) The original pictures of Figures 6.2(a) and (g) with impulse sampling; (b) zero-order hold applied to the pictures in (a). The visual system naturally introduces lowpass filtering with a cutoff frequency that decreases with distance. Thus, when viewed at a distance, the discontinuities in the mosaic in Figure 7.12(b) are smoothed; (c) result of applying a zero-order hold after impulse sampling with one-third the horizontal and vertical spacing used in (a) and (b).

Reconstruction of a signal from its samples using interpolation

- Linear interpolation: sometimes referred to as a first-order hold with $h(t)$ triangular, the associated transfer function is



$$H(j\omega) = \frac{1}{T}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega}{2}}\right]^2$$

$$H_0(j\omega) \propto \frac{\sin(\omega T/2)}{\omega/2}$$

Figure 7.13 Linear interpolation (first-order hold) as impulse-train sampling followed by convolution with a triangular impulse response: (a) system for sampling and reconstruction; (b) impulse train of samples; (c) impulse response representing a first-order hold;

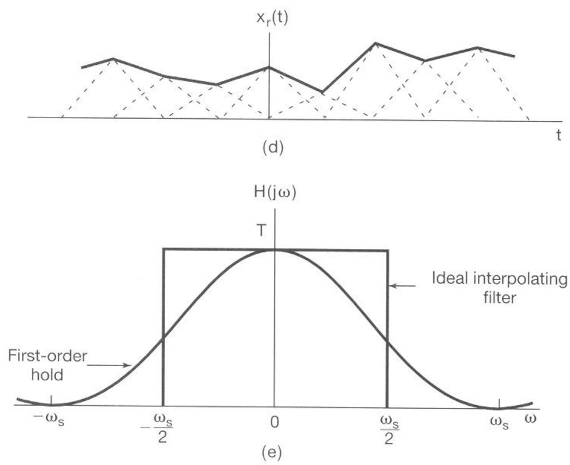


Figure 7.13 *Continued* (d) first-order hold applied to the sampled signal; (e) comparison of transfer function of ideal interpolating filter and first-order hold.

Figure 7.14 Result of applying a first-order hold rather than a zero-order hold after impulse sampling with one-third the horizontal and vertical spacing used in Figures 7.12(a) and (b).

7.3 The effect of undersampling: Aliasing

When $\omega_s < 2\omega_M$, the spectrum of $x(t)$, is no longer replicated in $X_p(j\omega)$ and thus is no longer recoverable by lowpass filtering. This effect, in which the individual terms in Eq. (7.6) overlap, is referred to as ***aliasing*** (混疊).

$$X_{p(j\omega)} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \quad (7.6)$$

The effect of undersampling: Aliasing

$$x_r(nT) = x(nT), \quad n = 0, \pm 1, \pm 2, \ldots\ldots \quad (7.13)$$

$$x(t) = \cos\omega_0 t \quad (7.14)$$

$\Rightarrow \omega_M = \omega_0$

(a) $\omega_0 = \frac{\omega_s}{6}$; $x_r(t) = \cos\omega_0 t = x(t)$, $\omega_s = 6\omega_0 > 2\omega_M$

(b) $\omega_0 = \frac{2\omega_s}{6}$; $x_r(t) = \cos\omega_0 t = x(t)$ $\omega_s = 3\omega_0 > 2\omega_M$

(c) $\omega_0 = \frac{4\omega_s}{6}$; $x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$, $\omega_s = 1.5\omega_0 < 2\omega_0$

(d) $\omega_0 = \frac{5\omega_s}{6}$; $x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$, $\omega_s = 1.2\omega_0 < 2\omega_0$

The effect of undersampling: Aliasing

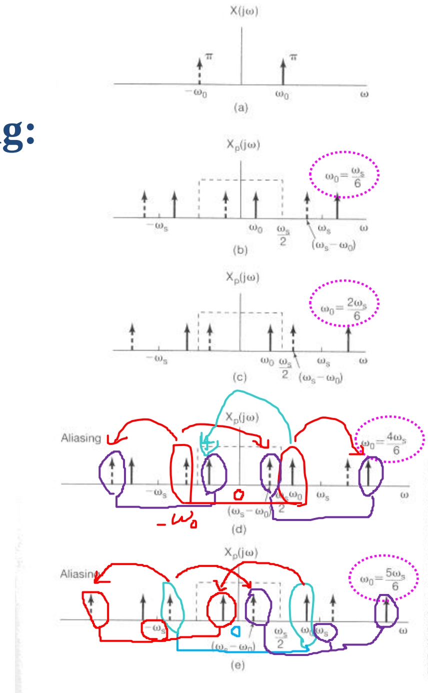
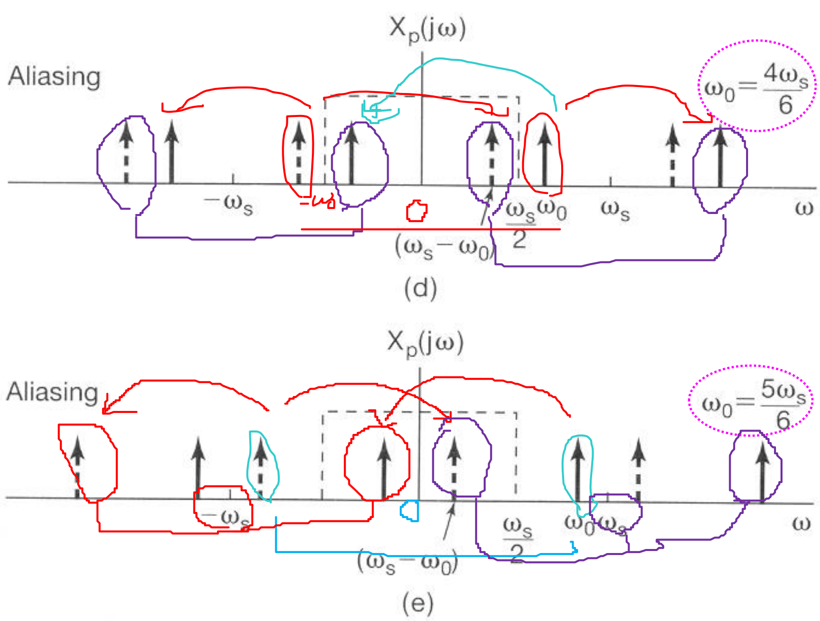


Figure 7.15 Effect in the frequency domain of oversampling and undersampling: (a) spectrum of original sinusoidal signal; (b), (c) spectrum of sampled signal with $\omega_s > 2\omega_0$; (d), (e) spectrum of sampled signal with $\omega_s < 2\omega_0$. As we increase ω_0 in moving from (b) through (d), the impulses drawn with solid lines move to the right, while the impulses drawn with dashed lines move to the left. In (d) and (e), these impulses have moved sufficiently that there is a change in the ones falling within the passband of the ideal lowpass filter.



$= \cos(\omega_0 t + \varphi) \quad (7.15)$

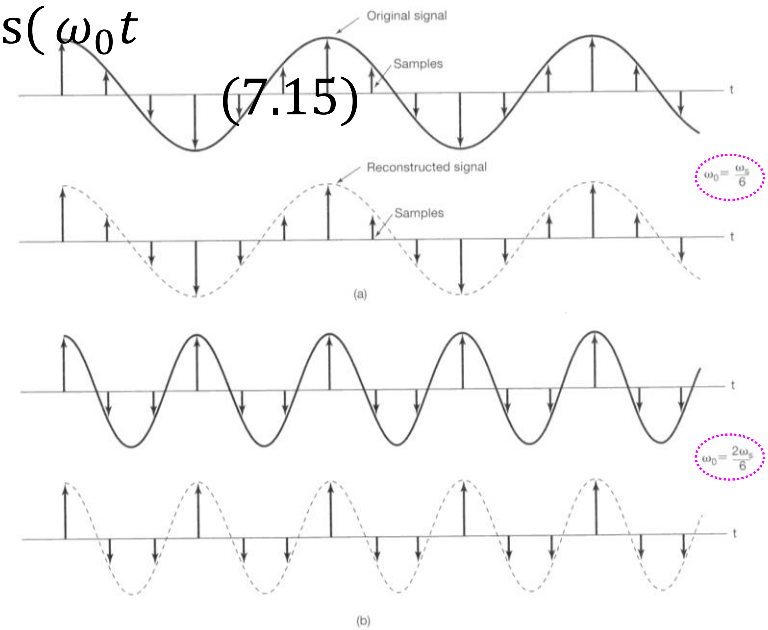


Figure 7.16 Effect of aliasing on a sinusoidal signal. For each of four values of ω_0, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_0 = \omega_s/6$; (b) $\omega_0 = 2\omega_s/6$; (c) $\omega_0 = 4\omega_s/6$; (d) $\omega_0 = 5\omega_s/6$. In (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.

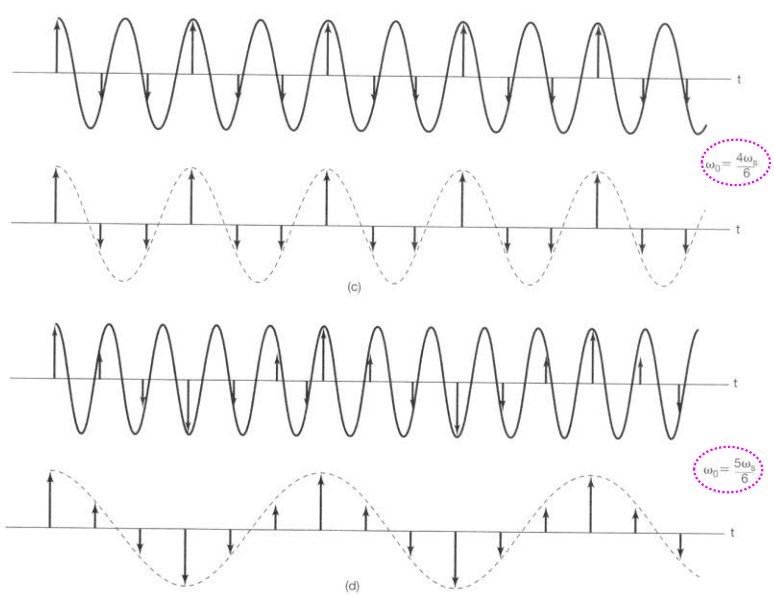


Figure 7.16 *Continued*

- The sampling theorem explicitly requires that the sampling frequency be ***greater*** than twice the highest frequency in the signal, rather than ***less than or equal to*** twice the highest frequency.
- Sampling a sinusoidal signal at exactly twice the highest frequency is ***not sufficient***.

Example 7.1
Consider the signal $x(t) = \cos(\frac{w_s}{2}t + \phi)$, and suppose that this signal is sampled, using impulse sampling, at exactly twice the frequency of the sinusoidal, w_s. If this impulse-sampled signal is applied as the input to an ideal lowpass filter with cutoff frequency $w_s/2$, the resulting output is

According to Prob. 7.39(舊版) 7.49(新版),

$$x_{r(t)} = \cos\emptyset \cos(\frac{\omega_s}{2}t)$$

Discussions:

(1) Perfect reconstruction of $x(t)$ occurs only in the case in which $\emptyset = 0$ or an integer multiple of 2π.

(2) Otherwise, the signal $x_r(t)$ does not equal $x(t)$.

(3) An extreme case, when $\emptyset = -\pi/2$, $x_r(t) = \sin(\frac{\omega_s}{2}t)$.

The values of the signal at integer multiples of the sample period $2\pi/\omega_s$ are zero.

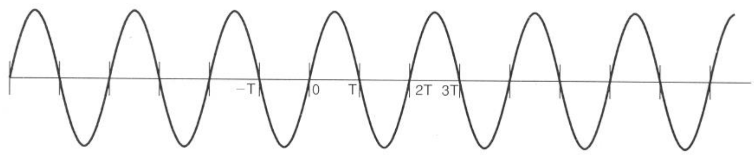


Figure 7.17 Sinusoidal signal for Example 7.1.

The effect of undersampling, whereby higher frequencies are reflected into lower frequencies, is the principle on which the stroboscopic effect (頻閃效應) is based.

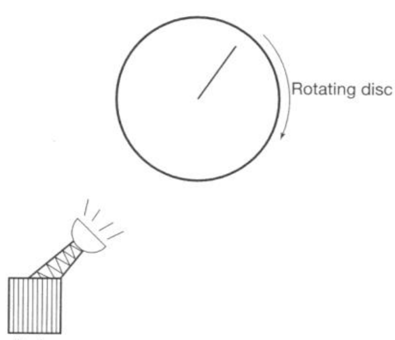


Figure 7.18 Strobe effect.

7.4 Discrete-Time Processing of Continuous-Time Signals

Through the process of periodic sampling with the sampling frequency consistent with the conditions of the sampling theorem, the continuous-time signal $x_c(t)$ is exactly represented by a sequence of instantaneous values $x_c(n\mathrm{T})$

$$x_d[n] = x_c(nT) \qquad (7.16)$$

Key concepts

1. sampling can be viewed as a C/D converter and reconstruction from sampling can be viewed as a D/C converter;

2. If a discrete - time system is equivalent to a continuous-time system, then their frequency responses $H_d(e^{j\Omega})$ and $H_c(j\omega)$ has the relations as in eq. (7.25) and eq (7.25a);

3. the trick to apply the input of a sinc function to convert a band-limited continuous operation (i.e., $H_c(j\omega) = 0$ for $|\omega| > \omega_c$, $\omega_c \leq \omega_s/2$) into an equivalent discrete operation (see Examples 7.2 and 7.3)

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7.4 Discrete-Time Processing of Continuous-Time Signals

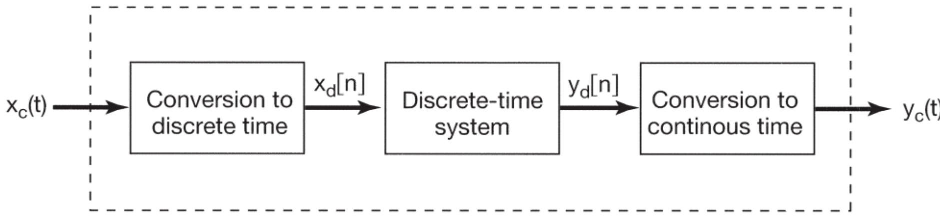


Figure 7.19 Discrete-time processing of continuous-time signals.

Figure 7.19 will be referred to as ***continuous-to-discrete-time conversion*** and will be abbreviated C/D. The reverse operation (D/C) corresponding to the third system in Figure 7.19

$$y_d[n] = y_c(nT)$$

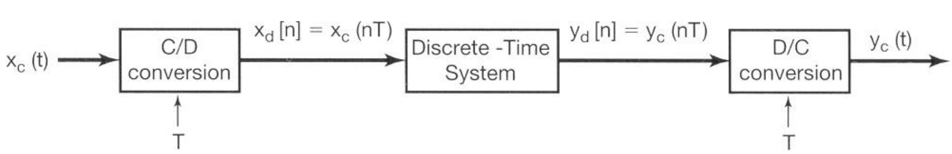
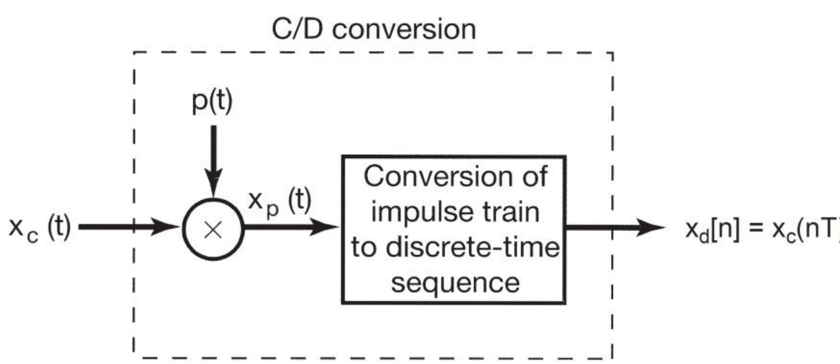


Figure 7.20 Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion. *T* represents the sampling period.

ω: continuous-time frequency variable

Ω: discrete-time frequency variable

The continuous-time Fourier transforms of $x_c(t)$ and $y_c(t)$ are $X_c(j\omega)$ and $Y_c(j\omega)$, respectively, while the discrete-time Fourier transform of $x_d[n]$ and $y_d[n]$ are $X_d(e^{j\Omega})$ and $Y_d(e^{j\Omega})$, respectively.



(a)

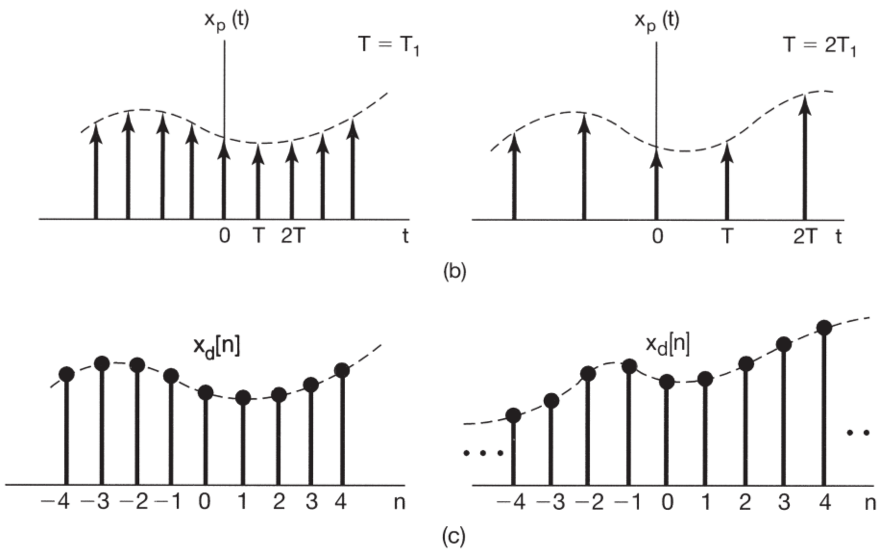


Figure 7.21 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.

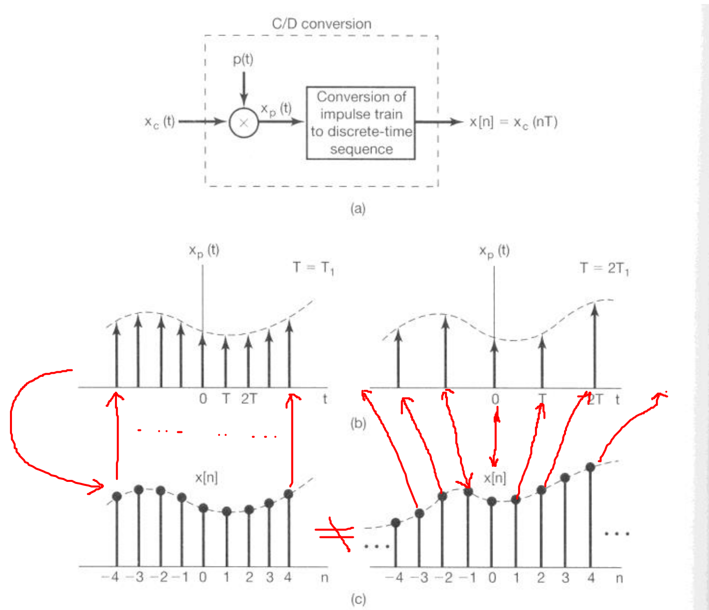


Figure 7.21 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

And since the transform of $\delta(t - nT)$ is $e^{-j\omega nT}$, it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega nT}$$

$$X_d\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n}$$

$X_d\left(e^{j\Omega}\right)$ and $X_p(j\omega)$ are related through

$$X_d\left(e^{j\Omega}\right) = X_p(j\Omega/T)$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\big(j(\omega - k\omega_s)\big) \qquad (7.6)$$

$$X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T) \qquad (7.23)$$

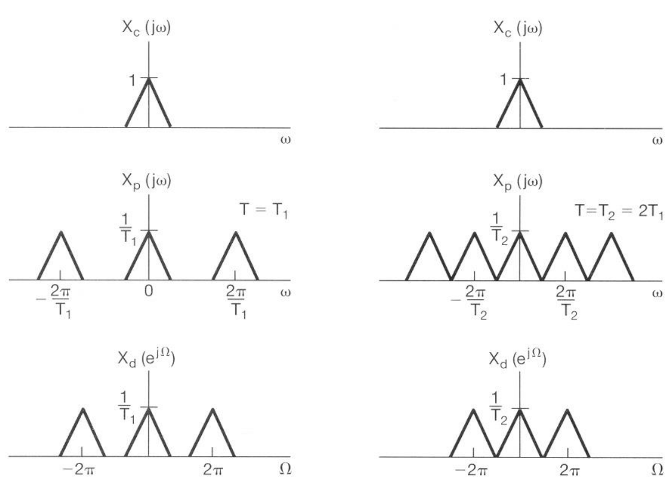


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

$X_d\left(e^{j\Omega}\right)$ is a frequency-scaled version of $X_p(j\omega)$ and, in particular, is periodic in Ω with period 2π.

- Recovery of the continuous-time signal $y_c(t)$ from this impulse train is then accomplished by means of lowpass filtering.

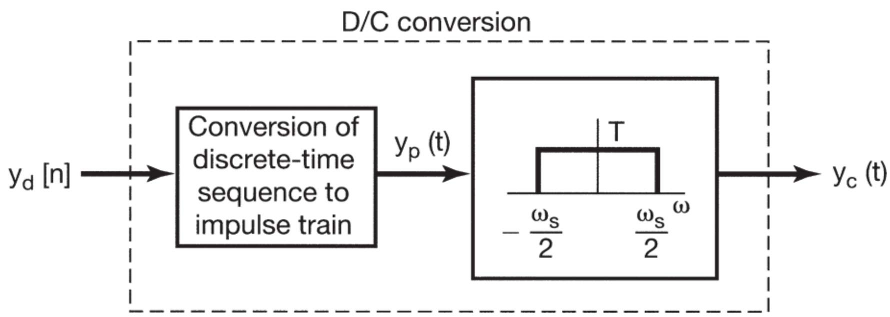


Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T}) \qquad (7.24)$$

The overall system of Fig. 7.24 is equivalent to a C-T LTI system with frequency response $H_c(j\omega)$ which is related to the D-T frequency response through

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega|<\omega_s/2 \\ 0, & |\omega|>\omega_s/2 \end{cases} \qquad (7.25)$$

In summary, the frequency response $H_d(e^{j\Omega})$ in Figure 7.24 can be derived from:

$$H_d(e^{j\Omega}) = H_c(j\Omega/T) \quad \text{for } |\Omega| < \pi, \qquad H_d(e^{j\Omega}) = H_d(e^{j(\Omega+2\pi)}). \tag{7.25a}$$

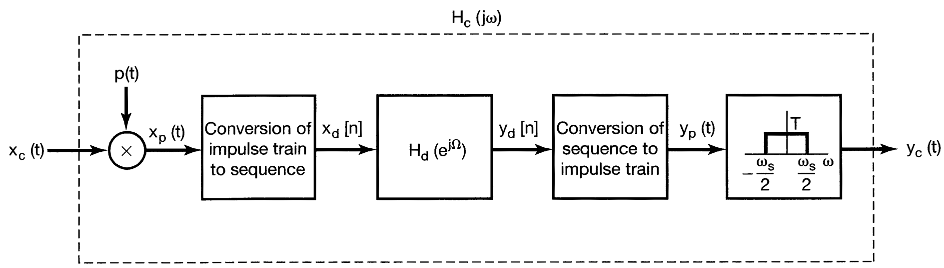


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.

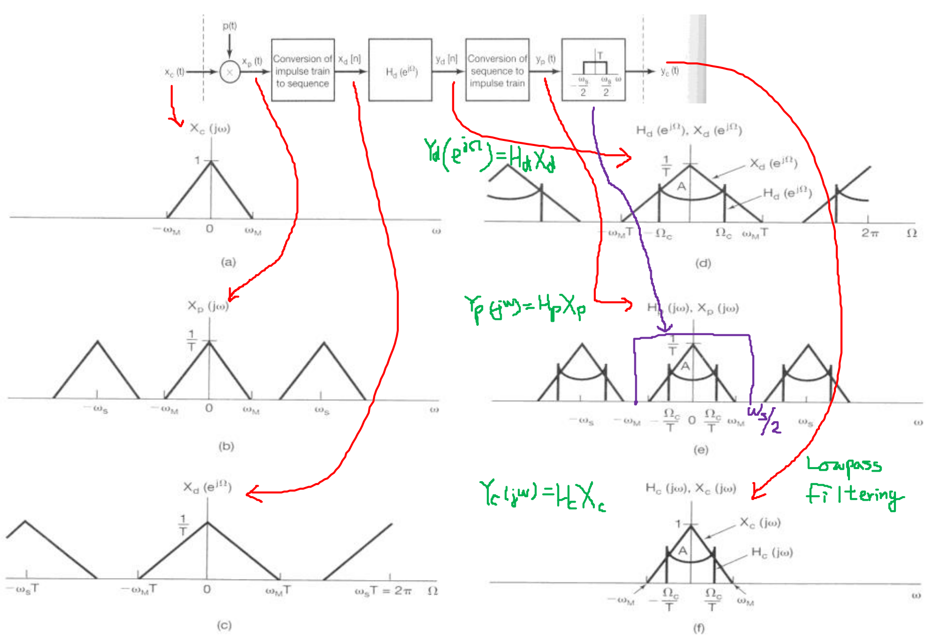


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.

Figure 7.26

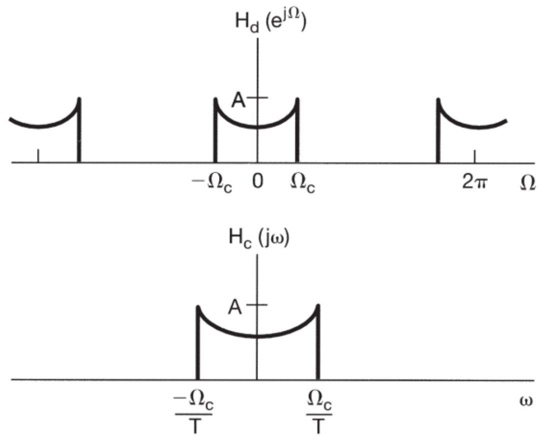


Figure 7.26 Discrete-time frequency response and the equivalent continuous-time frequency response for the system of Figure 7.24.

7.4.1 Digital Differentiator

Consider the discrete-time implementation of a continuous-time band-limited differentiating filter. the frequency response of a continuous-time differentiating filter is

$$H_c(j\omega) = j\omega, \tag{7.26}$$

with cutoff frequency ω_c is

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \tag{7.27}$$

the corresponding discrete-time transfer function is

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi, \tag{7.28}$$

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7.4.1 Digital differentiator

$$H_c(j\omega) = j\omega \tag{7.26}$$

$\omega_c = \frac{\omega_s}{2}$

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega|<\omega_c \\ 0, & |\omega|>\omega_c \end{cases} \tag{7.27}$$

$\omega_s = \frac{2\pi}{T}$

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \qquad |\Omega| < \pi \tag{7.28}$$

$\Omega = \omega_c \times T = \frac{\omega_s}{2} \times T = \pi$

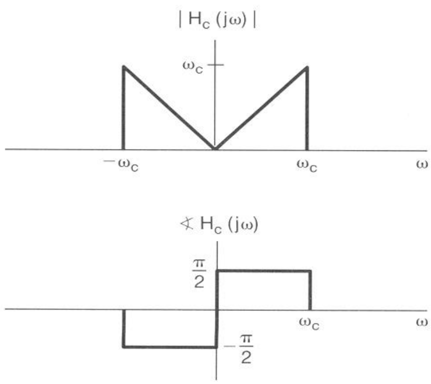

Figure 7.27 Frequency response of a continuous-time ideal band-limited differentiator $H_c(j\omega) = j\omega,\ |\omega| < \omega_c$.

Supplement

The inverse discrete-time Fourier transform of $H_d\,(e^{j\Omega})$ is

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{j}{2\pi T}\int_{-\pi}^{\pi} \Omega e^{j\Omega n}d\Omega$$
$$= \frac{\Omega}{2\pi Tn}\,e^{j\Omega n}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi Tn}\int_{-\pi}^{\pi} e^{j\Omega n}d\Omega = \frac{(-1)^n}{nT} \tag{7.28a}$$

when $n \neq 0$ and $h_d[0] = 0$. This is the discrete-time filter whose effect is equivalent to that of the continuous-time bandlimited differentiator.

P.593

Figure 7.28

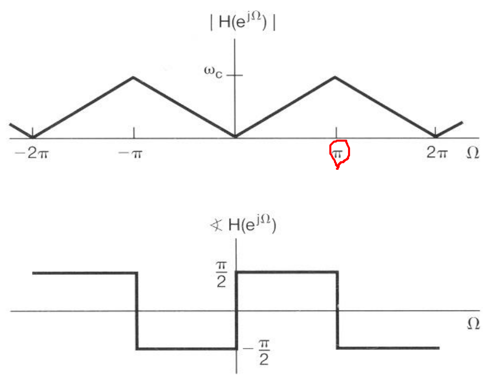

Figure 7.28 Frequency response of discrete-time filter used to implement a continuous-time band-limited differentiator.

[Example 7.2]

By considering the output of the digital differentiator for a continuous-time sinc input, we may conveniently determine the impulse response $h_d[n]$ of the discrete-time filter in the implementation of the digital differentiator. With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}, \tag{7.29}$$

where T is the sampling period. Then

$$X_c(j\omega) = \begin{cases} 1, & |\omega| < \pi/T \\ 0, & \text{otherwise} \end{cases},$$

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which is sufficiently band limited to ensure that sampling $x_c(t)$ at frequency $\omega_s = 2\pi/T$ does not give rise to any aliasing. It follows that the output of the digital differentiator is

$$y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}. \tag{7.30}$$

For $x_c(t)$ as given by eq. (7.29), the corresponding signal $x_d[n]$ in Figure 7.24 may be expressed as

$\frac{\sin(\pi n)}{\pi n T} = \frac{1}{T}\frac{\sin(\pi n)}{\pi n}$

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]. \tag{7.31}$$

$= \begin{cases} 0, & n \neq 0 \\ \frac{1}{T}, & n = 0 \end{cases}$

That is, for $n \neq 0$, $x_c(nT) = 0$, while $n=0$,

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$$x_d[0] = x_c(0) = \frac{1}{T}$$

- which can be verified by l'Hôpital's rule. We can similarly evaluate $y_d[n]$ in Figure 7.24 corresponding to $y_c(t)$ in eq. (7.30). Specifically,

$$y_d[n] = y_c(nT) = \begin{cases} \frac{(-1)^n}{nT^2}, & n \neq 0 \\ 0 & n = 0 \end{cases} \tag{7.32}$$

$(-1)^n = \cos(n\pi)$

which can be verified for $n \neq 0$ by direct substitution into eq. (7.30) and for $n = 0$ by application of l'Hôpital's rule.

- Thus when the input to the discrete-time filter given by eq. (7.28) is the scaled unit impulse in eq. (7.31), the resulting output is given by eq. (7.32). We then conclude that the impulse response of this filter is given by

$$h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

➔ **⊗Teacher Note**

Note that the impulse response $h_d[n]$ derived here is equivalent to that in eq. (7.28a).

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Supplement

- In Example 7.2, we apply a trick to find the discrete-time filter that has the effect equivalent to that of a continuous-time filter in the bandlimited case.

- That is:

(1) First, find the output $y_c(t)$ of the continuous-time system if the input $x_c(t)$ is the sinc function as in equation (7.29).
(2) Then, the impulse response of the equivalent discrete-time system is $h_d\,[n] = T\,y_c(nT\,)$.

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7.4.2 Half-Sample Delay

we require that the input and output of the overall system be related by

$$y_c(t) = x_c(t - \Delta) \qquad (7.33)$$

the equivalent continuous-time system to be implemented must be band limited. Therefore, we take

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}, \qquad (7.34)$$

With the sampling frequency ω_s taken as $\omega_s = 2\omega_c$, the corresponding discrete-time frequency response is

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi, \qquad (7.35)$$

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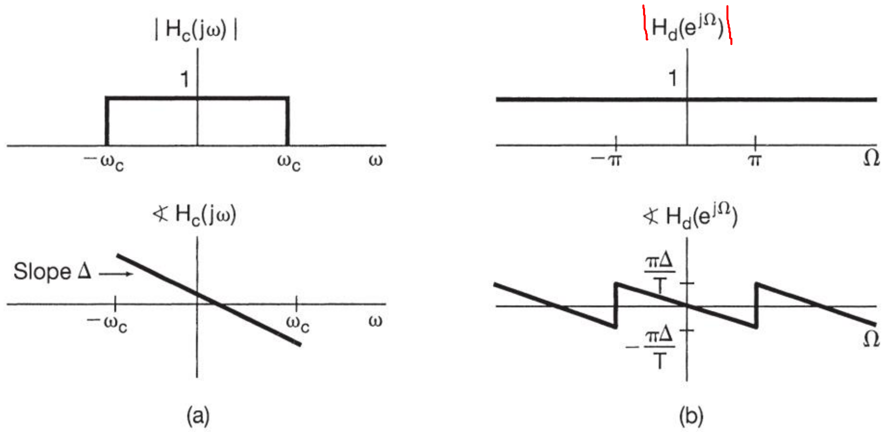

Figure 7.29 (a) Magnitude and phase of the frequency response for a continuous-time delay; (b) magnitude and phase of the frequency response for the corresponding discrete-time delay.

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Supplement

The inverse discrete-time Fourier transform of $H_d(e^{j\Omega})$ is

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n-\Delta/T)}d\Omega = \frac{\sin(\pi(n-\Delta/T))}{\pi(n-\Delta/T)}. \quad (7.35a)$$

This is the discrete-time filter whose effect is equivalent to that of the continuous-time bandlimited delay operation.

For Δ/T an integer, the sequence $y_d[n]$ is a delayed replica of $x_d[n]$; that is,

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]. \quad (7.36)$$

P.595-596

[Example 7.3]

The approach in Example 7.2 is also applicable to determine the impulse response $h_d[n]$ of the discrete-time filter in the half-sample delay system. With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}. \tag{7.37}$$

It follows from Example 7.2 that

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

Also, since there is no aliasing for the band-limited input in eq. (7.37), the output of the half-sample delay system is

$$y_c(t) = x_c(t - T/2) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)},$$

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$$y_c(t) = x_c(t - T/2) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)},$$

and the sequence $y_d[n]$ in Figure 7.24 is

$$y_d[n] = y_c(nT) = \frac{\sin(\pi(n - \frac{1}{2}))}{T\pi(n - \frac{1}{2})}.$$

We conclude that

$$h_d[n] = \frac{\sin(\pi(n - \frac{1}{2}))}{\pi(n - \frac{1}{2})}.$$

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$\Delta/T = \frac{1}{2}$

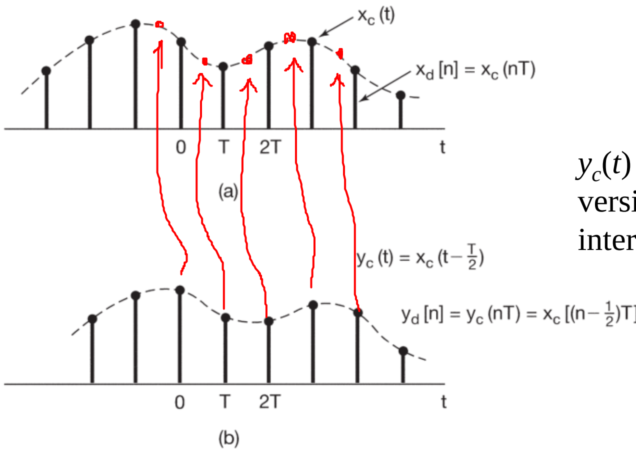

Figure 7.30 (a) Sequence of samples of a continuous-time signal $x_c(t)$; (b) sequence in (a) with a half-sample delay.

$y_c(t)$ is equal to samples of a shifted version of the band-limited interpolation of the sequence $x_d[n]$.

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Sec. 7.5 SAMPLING OF DISCRETE-TIME SIGNALS

Key concepts

(i) impulse-train sampling (equation 7.38) and its effect in the frequency domain [equation (7.42)];

(ii) how to reconstruct $x[n]$ after applying impulse-train sampling [see Fig. 7.33, equation (7.46), and equation (7.46a)];

(iii) Decimation (抽取) [equations (7.48) and (7.49)] and its effect in the frequency domain [equations (7.54) and (7.55)]; the process of decimation is called downsampling;

(iv) interpolation [equations (7.56)–(7.58)] and its effect in the frequency domain [equations (7.59)– (7.62) and Figure 7.37]; interpolation is also called upsampling

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7.5 Sampling of Discrete-Time Signals

- 7.5.1 Impulse-train sampling
- 7.5.2 Discrete-time decimation and interpolation

7.5 SAMPLING OF DISCRETE-TIME SIGNALS
7.5.1 Impulse-Train Sampling (1/3)

- The new sequence $x_p[n]$ resulting from the sampling process is equal to the original sequence $x[n]$ at integer multiples of the sampling period N and is zero at the intermediate samples; that is,

$$x_p[n] = \begin{cases} x[n], & \text{if } n = \text{an integer multiple of } N \\ 0, & \text{otherwise} \end{cases} \tag{7.38}$$

7.5.1 Impulse-Train Sampling (2/3)

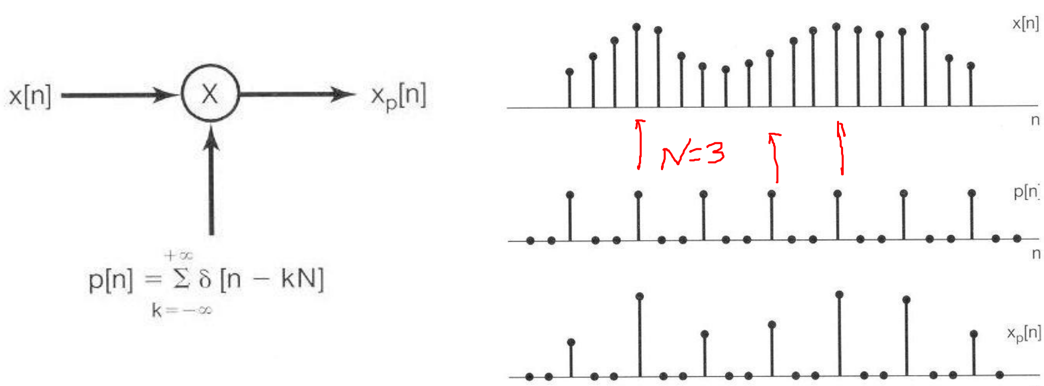
- The effect in the frequency domain of discrete-time sampling is seen by using the multiplication property developed in Section 5.5. Thus, with

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n-kN] \qquad (7.39)$$

Figure 7.31 Discrete-time sampling

$X_p = X * P$

$$X_p(e^{j\varpi}) = \frac{1}{2\pi}\int_{2\pi} \mathrm{P}(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$



7.5.1 Impulse-Train Sampling (3/3)

- AS in Example 5.6, the Fourier transform of the sampling sequence $p[n]$ is

$$P(\mathrm{e}^{\mathrm{j}\omega}) = \frac{2\pi}{N}\sum_{K=-\infty}^{+\infty} \delta(\omega - K\omega_s)$$

where ω_s, the sampling frequency, equals $2\pi/N$.

- Combining Eqs. (7.40) and (7.41), we have

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$

Sampling of a discrete-time signal: the new sequence $x_p[n]$ resulting from the sampling process is equal to the original sequence $x[n]$ at integer multiples of the sampling period N and is zero at the intermediate samples;

$$x_p[n] = \begin{cases} x[n], & \text{if } n = \text{an integer multiples of } N \\ 0, & \text{otherwise} \end{cases}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN]$$

$$X_p\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)\mathrm{X}\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$

$$P\left(e^{j\omega}\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$X_p\left(e^{j\omega}\right) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$

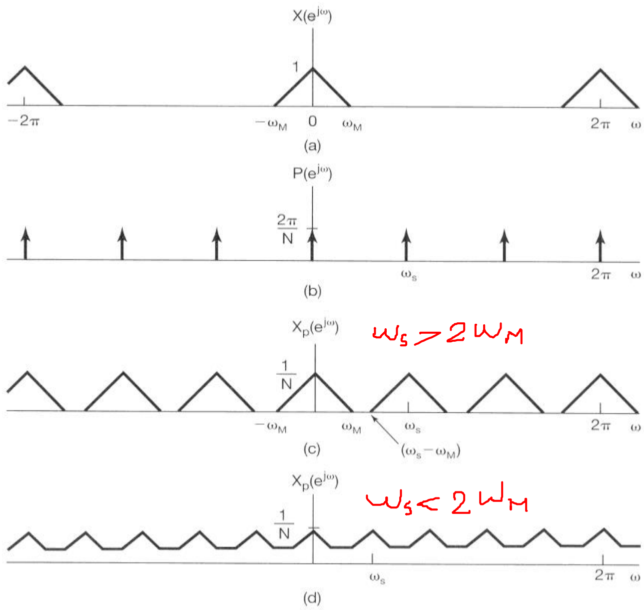
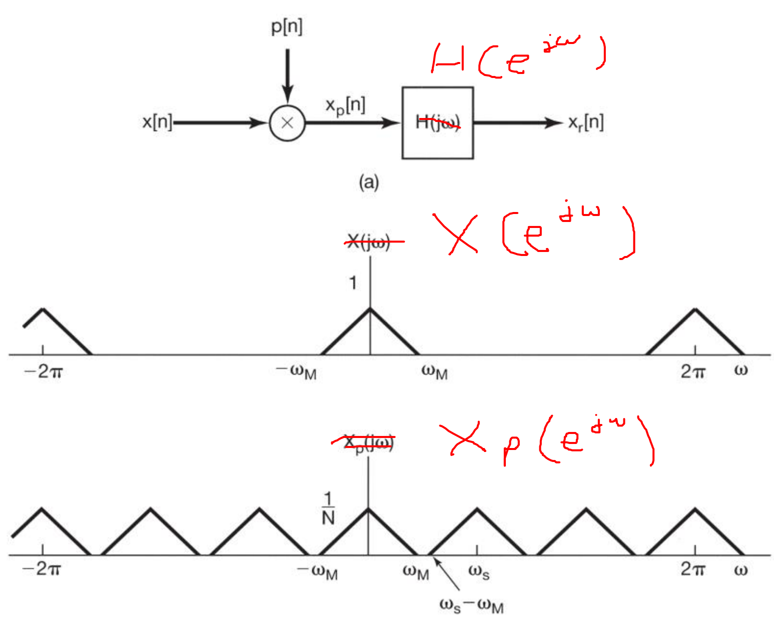


Figure 7.32 Effect in the frequency domain of impulse-train sampling of a discrete-time signal: (a) spectrum of original signal; (b) spectrum of sampling sequence; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$. Note that aliasing occurs.



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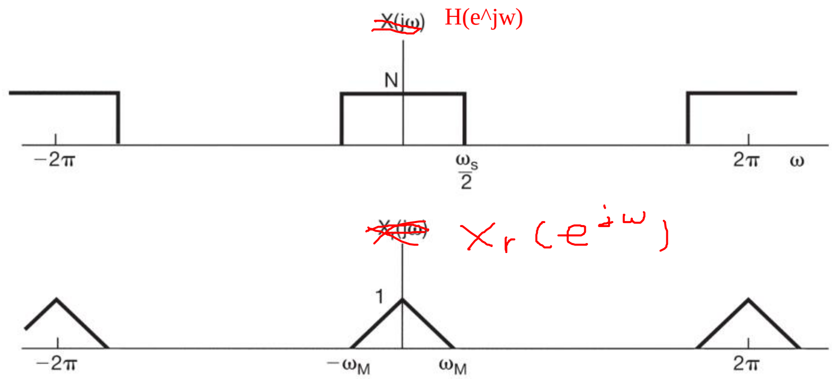

(b)

Figure 7.33 Exact recovery of a discrete-time signal from its samples using an ideal lowpass filter: (a) block diagram for sampling and reconstruction of a band-limited signal from its samples; (b) spectrum of the signal $x[n]$; (c) spectrum of $x_p[n]$; (d) frequency response of an ideal lowpass filter with cutoff frequency $\omega_s/2$; (e) spectrum of the reconstructed signal $x_r[n]$. For the example depicted here $\omega_s > 2\omega_M$ so that no aliasing occurs and consequently $x_r[n] = x[n]$.

P.600

[Example 7.4]

Consider a sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ has the property that

$$X(e^{j\omega}) = 0 \quad \text{for} \quad 2\pi/9 \leq |\omega| \leq \pi.$$

To determine the lowest rate at which $x[n]$ may be sampled without the possibility of aliasing, we must find the largest N such that

$$\frac{2\pi}{N} \geq 2\left(\frac{2\pi}{9}\right) \Longrightarrow N \leq 9/2.$$

We conclude that $N_{max} = 4$, and the corresponding sampling frequency is $2\pi/4 = \pi/2$.

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With $h[n]$ denoting the impulse response of the lowpass filter, we have

$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}. \tag{7.44}$$

then

$$x_r[n] = x_p[n] * h[n], \tag{7.45}$$

or equivalently,

$$\boxed{x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin\omega_c(n-kN)}{\omega_c(n-kN)}.} \tag{7.46}$$

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n-kN], \tag{7.47}$$

where $h_r[n]$ is the impulse response of the interpolation filter.

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7.5.2 Discrete-Time Decimation and Interpolation

the sampled sequence is typically replaced by a new sequence $x_b[n]$, which is simply every Nth value of $x_p[n]$; that is,

$$x_b[n] = x_p[nN]. \tag{7.48}$$

Also, equivalently,

$$x_b[n] = x[nN], \tag{7.49}$$

P.602

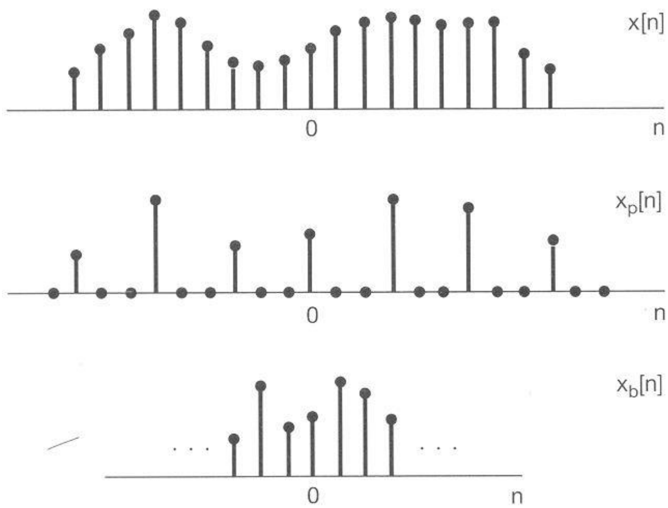


Figure 7.34: Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation

In between the sampling instants, $x_p[n]$ is known to be zero. Therefore, it is inefficient to represent, transmit, or store the sampled sequence. The sampled sequence $x_p[n]$ is typically replaced by a new sequence $x_b[n]= x_p[nN] = x[nN]$.

Decimation: the operation of extracting every *N*th sample.

$$X_{b(}e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} x_p[kN]\, e^{-j\omega k}$$

Let $n = kN$, $$Xb_{(}e^{j\omega}) = \sum_{n=kN} x_p[n]e^{-j\omega n/N}$$

$$X_{b(}e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} = Xp\left(e^{\frac{j\omega}{N}}\right)$$

Since $x_p[n]$=0 when *n* is not an integer multiple of *N*.

- The spectra for the sampled sequence and the decimated sequence differ only in a frequency scaling or normalization.
- The effect of decimation is to spread the spectrum of the original sequence over a larger portion of the frequency band.

Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation

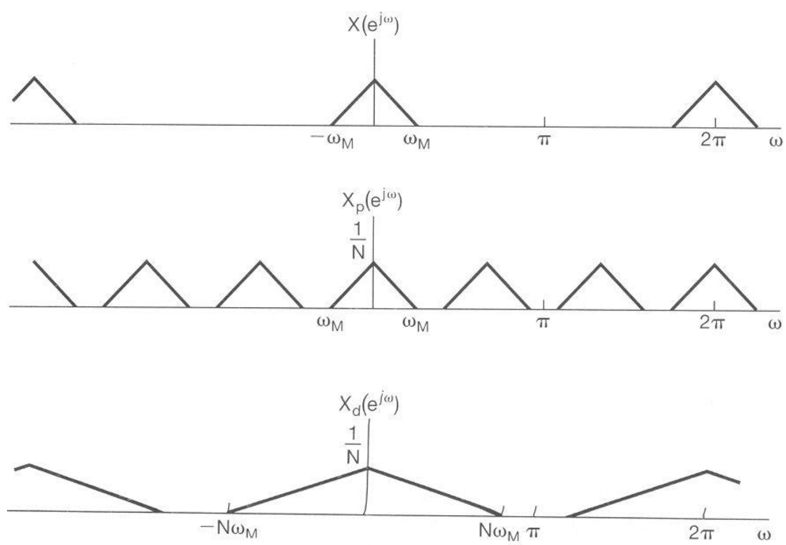
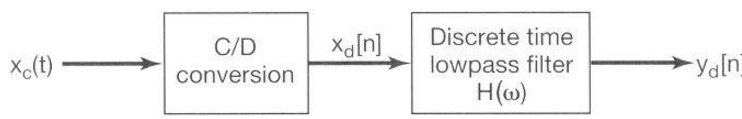


Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation.

Figure 7.36 (1/2)



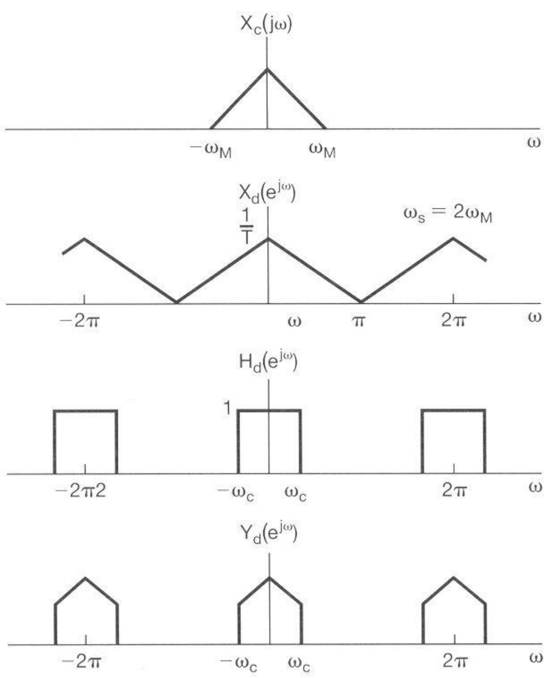
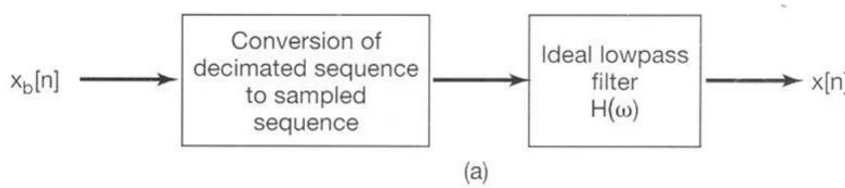


Figure 7.36 Continuous-time signal that was originally sampled at the Nyquist rate. After discrete-time filtering, the resulting sequence can be further downsampled. Here $X_c(j\omega)$ is the continuous-time Fourier transform of $x_c(t)$, $X_d(e^{j\omega})$ and $Y_d(e^{j\omega})$ are the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ respectively, and $H_d(e^{j\omega})$ is the frequency response of the discrete-time lowpass filter depicted in the block diagram.

Figure 7.37 (1/2)



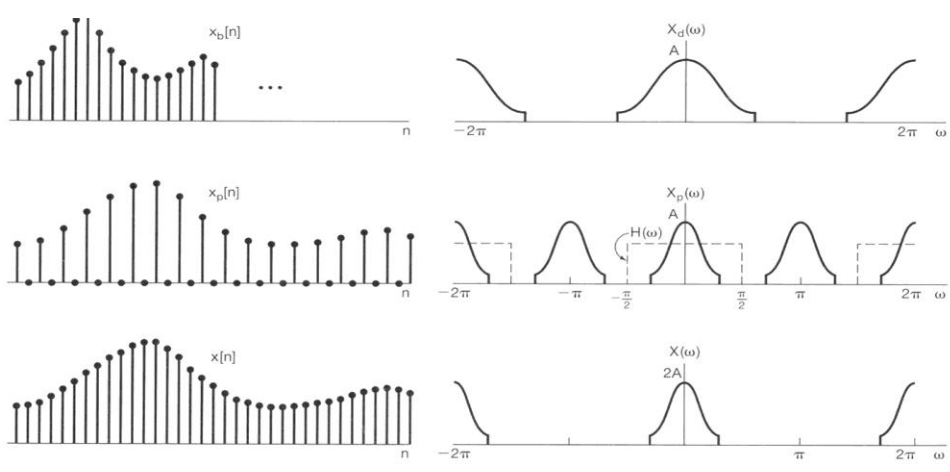


Figure 7.37 Unsampling: (a) overall system; (b) associated sequences and spectra for upsampling by a factor of 2.

Example 7.5

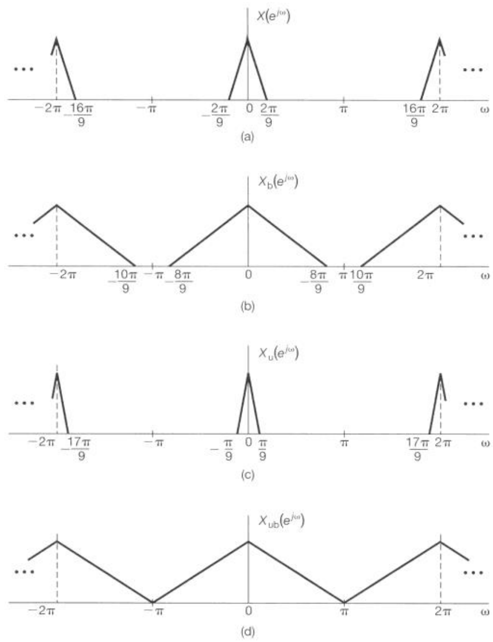


Figure 7.38 Spectra associated with Example 7.5. (a) Spectrum of $x[n]$; (b) spectrum after downsampling by 4; (c) spectrum after upsampling $x[n]$ by a factor of 2; (d) spectrum after upsampling $x[n]$ by 2 and then downsampling by 9.

Homework

- Homework(新版)
- 7.3, 7.6, 7.8, 7.10, 7.14, 7.19
- 7.19舊版沒有

- Homework(舊版)
- 7.3, 7.6, 7.9, 7.11, 7.15

Determine the frequency response of the filter whose impulse response is $h[2n]$.

7.19. Consider the system shown in Figure P7.19(a) for filtering a CT signal using a DT filter. Here $x[n] = x_c(nT)$, $y[n] = y_c(nT)$, and $L_T(j\omega)$ is an ideal low-pass filter with cutoff frequency π/T and gain T.

If $X_c(j\omega)$ and $H(e^{j\Omega})$ are as shown in Figure P7.19(b), and if T is chosen to provide sampling of $x_c(t)$ at the Nyquist frequency, sketch $X_p(j\omega)$, $X(e^{j\Omega})$, $Y(e^{j\Omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.

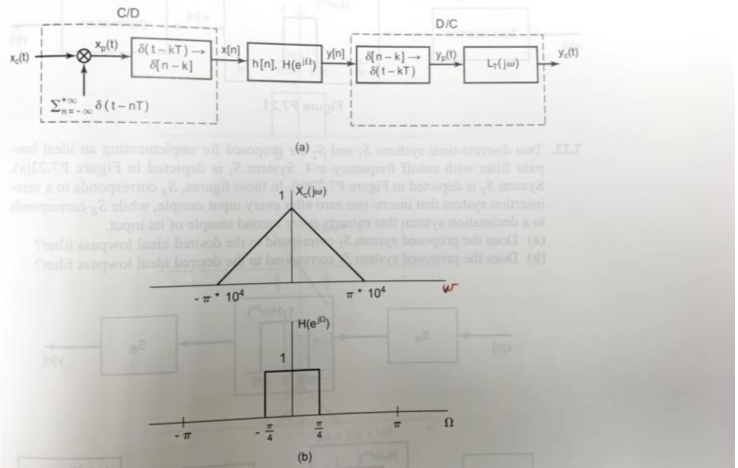


Figure P7.19